Column Buckling

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Stability and Buckling

- **Stability** is characterized as the ability of a structure to maintain its (stable) equilibrium under working conditions.

- **Buckling** is the behavior of a structure losing its equilibrium under working conditions. This is another type of failure criterion, in addition to strength (fracture/yielding), stiffness (deformation) and fatigue criteria.

- Buckling occurs suddenly and results in catastrophic accident.
Conventional Design of Columns

• In the design of columns, cross-sectional area is selected such that
  - allowable stress is not exceeded
    \[ \sigma = \frac{P}{A} \leq [\sigma] \]
  - deformation falls within specifications
    \[ \varepsilon = \frac{\Delta L}{L} = \frac{P}{AE} \leq [\varepsilon] \]

• After these design calculations, many discover that the column is unstable under loading and that it suddenly becomes sharply curved or buckles.
Consider an axially loaded beam. After a small perturbation, the system reaches a neutral equilibrium configuration such that

\[
\frac{d^2 w}{dx^2} + \frac{P_{cr}}{EI} w = 0
\]

\[
\Rightarrow w'' + k^2 w = 0, \quad k^2 = \frac{P_{cr}}{EI}
\]

\[
\Rightarrow w = A \sin kx + B \cos kx
\]

\[
0 = w(0) = w(L)
\]

\[
B = 0;
\]

\[
kL = n\pi \Rightarrow P_{cr} = \frac{EIn^2\pi^2}{L^2}
\]
Buckling Modes

\[
P_{cr} = \frac{EI\pi^2}{L^2}; \quad P_{cr} = \frac{4EI\pi^2}{L^2}; \quad P_{cr} = \frac{9EI\pi^2}{L^2}
\]
Cantilevered Columns

- A column with one fixed and one free end, will behave as the upper-half of a pin-connected column.

- The critical loading is calculated from Euler’s formula,

\[
P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 EI}{4L^2}
\]

\[
\sigma_{cr} = \frac{\pi^2 E}{(L_e/i_r)^2} = \frac{\pi^2 E}{4(L/i_r)^2}
\]

\[L_e = 2L = \text{equivalent length}\]
Columns with Two Fixed Ends

- The symmetry of the supports and of the loading requires that the shear at $C$ and the horizontal reactions at both ends be zero.
- The equation of the deflection curve involves sine and cosine functions.
- Point $D$ must be a point of inflection, where the bending moment is zero.
- It follows that the portion $DE$ of the column must behave as a pin ended column.

\[
L_e = 0.5L \quad \Rightarrow \quad P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{4\pi^2 EI}{L^2}
\]
Columns with One Fixed End and One Free End

- The differential equation

\[
\frac{d^2 w}{dx^2} = -\left(\frac{Pw - Vx}{EI}\right)
\]

\[
\Rightarrow \frac{d^2 w}{dx^2} + \frac{P}{EI} w = \frac{Vx}{EI}
\]

\[
\Rightarrow w = A \sin kx + B \cos kx + \frac{Vx}{P}, \quad k^2 = \frac{P}{EI}
\]

\[
\begin{align*}
0 &= w(0) = w(L) \quad \Rightarrow B = 0; \quad A \sin kL = -\frac{VL}{P} \\
0 &= w'(L) \quad \Rightarrow Ak \cos kL = -\frac{V}{P} \\
\Rightarrow \tan kL &= kL \quad \Rightarrow k^2 = 20.19/L^2 \\
\Rightarrow P_{cr} &= EI k^2 = \frac{20.19EI}{L^2} \approx \frac{\pi^2 EI}{(0.699L)^2}
\end{align*}
\]

- Equivalent length: \( L_e \approx 0.7L \)
Extension of Euler’s Formula

(a) One fixed end, one free end

(b) Both ends pinned

(c) One fixed end, one pinned end

(d) Both ends fixed

\[ P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 EI}{(\mu L)^2}; \]

\[ \sigma_{cr} = \frac{\pi^2 E}{(L_e/i_r)^2} = \frac{\pi^2 E}{(\mu L/i_r)^2} \]

\[ L_e = \text{equivalent length}; \quad \mu = \text{length coefficient} \]
Sample Problem

• Assuming the same material and cross-sectional dimension, which of the following four columns is most susceptible to buckling?

\[ P_{cr} = \frac{\pi^2 EI}{(\mu l)^2} \quad \text{(Euler Formula)} \]
Sample Problem

- For the truss shown, \( F, \beta, \) and \( L_{AC} \) are given. Find \( 0 < \theta < \pi/2 \), under which column \( AB \) and \( BC \) reach critical stability simultaneously.

- Solution:

  1. Euler’s equation for column \( AB \):

   \[
   F_{AB} = F \cos \theta = \frac{\pi^2 EI}{l_{AB}^2} = \frac{\pi^2 EI}{l^2 \cos^2 \beta}
   \]

  2. Euler’s equation for column \( BC \):

   \[
   F_{BC} = F \sin \theta = \frac{\pi^2 EI}{l_{CB}^2} = \frac{\pi^2 EI}{l^2 \sin^2 \beta}
   \]

  3. If column \( AB \) & \( BC \) reach critical stability simultaneously:

   \[
   \left\{ \begin{aligned}
   F \cos \theta &= \frac{\pi^2 EI}{l^2 \cos^2 \beta} \\
   F \sin \theta &= \frac{\pi^2 EI}{l^2 \sin^2 \beta}
   \end{aligned} \right. \quad \Rightarrow \cot \theta = \tan^2 \beta
   \]
Sample Problem

- Which of the following best describes the relationship between critical $P_1$ and $P_2$? (a) $P_1 = P_2$; (b) $P_1 < P_2$; (c) $P_1 > P_2$; (d) not sure.

- Solution: (a) $F_{NAD} = \sqrt{2}P_1 = \frac{\pi^2 EI}{(\sqrt{2}a)^2} \implies P_1 = \frac{1}{2\sqrt{2}} \frac{\pi^2 EI}{a^2}$

(b) $F_{NAB} = P_2 = \frac{\pi^2 EI}{a^2} \implies P_2 = \frac{\pi^2 EI}{a^2}$
For the column shown, find the relationship between the critical loads corresponding to cross-sections described by (a), (b) and (c).

Solution:

\[
\begin{align*}
    P_{cr-a} &= \frac{\pi^2 EI_a}{(\mu l)^2} = \frac{\pi^2 E \pi (2r)^4}{64 (\mu l)^2} = \frac{\pi^2 E \pi r^4}{4 (\mu l)^2} \\
    P_{cr-b} &= \frac{\pi^2 EI_b}{(\mu l)^2} = \frac{\pi^2 E \pi r^4}{64 (\mu l)^2} \\
    P_{cr-c} &= \frac{\pi^2 EI_c}{(\mu l)^2} = \frac{\pi^2 E (r \sqrt{\pi})^4}{12 (\mu l)^2} = \frac{\pi^2 E \pi^2 r^4}{12 (\mu l)^2}
\end{align*}
\]

\[
\begin{align*}
    P_{cr-a} : P_{cr-b} : P_{cr-c} &= 1 : 1 : \pi \\
    &= \frac{1}{4} : \frac{1}{64} : \frac{\pi}{12} \\
    &= 1 : \frac{1}{16} : \frac{\pi}{3}
\end{align*}
\]
Buckling in Orthogonal Planes

• For the column shown, find the relationship between the critical loads corresponding to buckling in $x$-$y$ and $x$-$z$ plane respectively.

• Solution:

$$P_{cr,z} = \frac{\pi^2 EI_z}{(\mu_z l)^2}$$

$$P_{cr,y} = \frac{\pi^2 EI_y}{(\mu_y l)^2}$$

$$\frac{P_{cr,z}}{P_{cr,y}} = \frac{I_z}{I_y} = \frac{b(2b)^3/12}{2bb^3/12} = 4$$

• What if the cross-section is increased to $2b \times 2b$?
Buckling in Orthogonal Planes

- Buckling in $x$-$z$:

$$\mu_y = 1 \quad P_{cr.y} = \frac{\pi^2 EI_y}{L^2}$$

- Buckling in $x$-$y$:

$$\mu_z = 0.5 \quad P_{cr.z} = \frac{\pi^2 EI_z}{(0.5L)^2}$$
Ways to Improve Column Stability

• Selection of materials.
• Decrease effective column length ($\mu L$).
• Increase moment of inertia for a given cross-sectional area.

\[ P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 EI}{(\mu L)^2} \]

\[ \sigma_{cr} = \frac{\pi^2 E}{(\mu L/i_r)^2} = \frac{\pi^2 E}{\lambda^2} \]

• Coordination of end conditions and moment of inertia in orthogonal planes: $\mu_z/i_{r,z} = \mu_y/i_{r,y}$

• For a group of columns, make each one equally stable (avoid the last straw).
Applicability of Euler’s Formula

• Critical buckling stress

\[
P_{cr} = \frac{\pi^2 EI}{(\mu L)^2} \quad \Rightarrow \quad \sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E (A i_r^2)}{(\mu L)^2 A} = \frac{\pi^2 E}{(\mu L / i_r)^2} = \frac{\pi^2 E}{\lambda^2}
\]

\[
\lambda = \mu L / i_r = \text{slenderness ratio}
\]

• Applicability of Euler’s formula

\[
\sigma_{cr} = \frac{\pi^2 E}{\lambda^2} \leq \sigma_p \quad \Rightarrow \quad \lambda \geq \lambda_p = \pi \sqrt{\frac{E}{\sigma_p}} \quad \sigma_p: \text{proportion limit}; \quad \lambda_p: \text{critical slenderness ratio}.
\]

Q235:

\[
\lambda_p = \pi \sqrt{\frac{E}{\sigma_p}} = \pi \sqrt{\frac{206 \text{ GPa}}{200 \text{ MPa}}} = 100
\]

\[
(\sigma_Y = 235 \text{ MPa})
\]
Previous analyses assumed stresses below the proportional limit and initially straight, homogeneous columns.

Experimental data demonstrate:
- For large $L_e / i_r$, $\sigma_{cr}$ follows Euler’s formula and depends upon $E$ but not $\sigma_Y$.
- For small $L_e / i_r$, $\sigma_{cr}$ is determined by the yield strength $\sigma_Y$ and not $E$.
- For intermediate $L_e / i_r$, $\sigma_{cr}$ depends on both $\sigma_Y$ and $E$. 

\[ \sigma_{cr} = \pi^2 E / \lambda^2 \]

Empirical formulas for intermediate columns
Critical Stress of Columns

1. Long columns: \( \lambda \geq \lambda_P \left( \sigma_{cr} = \frac{\pi^2 E}{\lambda^2} \leq \sigma_p = \frac{\pi^2 E}{\lambda_P^2} \right) \)

2. Intermediate columns:
   \[ \lambda_Y = \frac{a - \sigma_Y}{b} < \lambda < \lambda_P \left( \sigma_p < \sigma_{cr} = a - b\lambda < \sigma_Y \right) \]

3. Short columns: \( \lambda < \lambda_Y \left( \sigma_{cr} > \sigma_Y \right) \)
Sample Problem

- For the wood column shown, $E = 10$ GPa, $\sigma_p = 9$ MPa, $\sigma_Y = 13$ MPa, $\sigma_{cr} = 28.7 - 0.19\lambda$ for intermediate columns. Find the critical loads for rectangular cross-sections: (1) $h = 120$ mm, $b = 90$ mm; (2) $h = b = 104$ mm

- Solution:
  \[
  \lambda_p = \sqrt{\frac{\pi^2 E}{\sigma_p}} = 104.7
  \]
  \[
  \lambda_Y = \frac{a - \sigma_Y}{b} = 82.6
  \]

1. $\lambda = \frac{\mu l}{i_r} = 115.4$

   $\therefore \lambda > \lambda_p$ $\therefore$ Slender column.

   \[
   P_{cr} = \frac{\pi^2 EI}{(\mu l)^2} = 79.9$ kN
   \]

2. $\lambda = 100$

   $\therefore$ Intermediate column.

   \[
   P_{cr} = \sigma_{cr}A = (a - b\lambda)A = 104.9$ kN
   \]
Sample Problem

- For the composite beam and column structure shown, $E_{AB} = E_{BD} = E$, $d_{AB} = d_{BD} = d$, $L:d = 30$, $\lambda_{P,BD} = 100$. Find $P$, under which $BD$ reaches the critical condition.

- Solution:

  $BD : \lambda = \mu L \frac{i}{d/4} = 120 > \lambda_p$

  $F_{cr,BD} = \frac{\pi^2 EI}{L^2}$

- Deformation compatibility:

  $\frac{5PL^3}{6EI} - \frac{8F_{BD}L^3}{3EI} = \frac{F_{BD}L}{EA}$

  $\Rightarrow P = \frac{\pi^3 Ed^2}{18000}$
Design of Columns

1. Method of safety factor

\[ F \leq \frac{F_{cr}}{n_{st}} = [F_{st}] \]

\[ \sigma = \frac{F}{A} \leq \frac{\sigma_{cr}}{n_{st}} = [\sigma_{st}] \]

\( n_{st} \): safety factor

\([F_{st}]\): allowable load

\([\sigma_{st}]\): allowable stress

2. Method of discount factor

\[ [\sigma_{st}] = \varphi [\sigma] \]

\[ \sigma = \frac{F}{A} \leq \varphi [\sigma] \]

\( \varphi = \varphi(\lambda) \): discount/stability factor

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- Stability analysis of columns
  - Stability check
  - Cross-section design
  - Allowable load/stress
Sample Problem

• A Q275 steel column is cylindrically pinned at both ends. \( \lambda_p = 96, \ \sigma_Y = 275 \text{ MPa}, \ \sigma_{cr} = 280 - 0.00872\lambda^2 \) for intermediate columns. Analyze the column stability for \( F = 60 \text{ kN} \) and \( n_{st} = 3.5 \).

- Solution: \( \lambda_p = 96, \ \lambda_Y = \sqrt{\frac{a - \sigma_Y}{b}} \approx 24 \)

- In x-y plane, both ends are pinned.

\[
i_z = \sqrt{\frac{I_z}{A}} = \frac{h}{2\sqrt{3}} = 12.99 \text{ mm}, \quad \lambda_z = \frac{\mu_z l_z}{i_z} = \frac{1 \times 800}{12.99} = 61.9
\]
- In $x$-$z$ plane, both ends are fixed.

$$i_y = \sqrt{\frac{I_y}{A}} = \frac{b}{2\sqrt{3}} = 5.77\, \text{mm}, \quad \lambda_y = \frac{\mu_y l_y}{i_y} = \frac{0.5 \times 770}{5.77} = 66.7$$

$$\Rightarrow (\lambda_y = 24) < (\lambda_z = 61.9) < (\lambda_y = 66.7) < (\lambda_p = 96)$$

- The column buckling first happens in the $x$-$z$ plane.

$$\sigma_{cr} = 280 - 0.00872 \lambda_y^2 = 241 \, \text{MPa}$$

$$F_{cr} = \sigma_{cr} A = \left(241 \times 10^6 \right) \left(20 \times 45 \times 10^{-6} \right) = 217 \, \text{kN}$$

$$[F] = F_{cr} / n = 62 \, \text{kN}$$

$$\Rightarrow (F = 60 \, \text{kN}) < ([F] = 62 \, \text{kN})$$

- The column is stable.
Eccentric Loading: The Secant Formula

- Eccentric loading is equivalent to a centric load and a moment.
- Bending occurs for any nonzero eccentricity. Question of buckling becomes whether the resulting deflection is excessive.
- The deflection become infinite when $P = P_{cr}$

\[
\frac{d^2w}{dx^2} = -\frac{Pw - Pe}{EI} \quad \Rightarrow \quad \frac{d^2w}{dx^2} + \frac{P}{EI}w = -\frac{Pe}{EI}
\]

\[
w = A \sin kx + B \cos kx - e, \quad k^2 = \frac{P}{EI}
\]

\[
0 = w(0) = w(L)
\]

\[
B = e; A \sin kL = e(1 - \cos kL) \quad \Rightarrow \quad A = e \tan \left(\frac{kL}{2}\right)
\]

\[
w = e \left(\tan \left(\frac{kL}{2}\right) \sin kx + \cos kx - 1\right)
\]

\[
w_{max} = w\left(\frac{L}{2}\right) = e \left(\tan \left(\frac{kL}{2}\right) \sin \left(\frac{kL}{2}\right) + \cos \left(\frac{kL}{2}\right) - 1\right)
\]

\[
= e \left(\sec \left(\frac{kL}{2}\right) - 1\right) = e \left[\sec \left(\frac{L}{2} \sqrt{\frac{P}{EI}}\right) - 1\right]
\]

\[
= e \left[\sec \left(\frac{\pi}{2} \sqrt{\frac{P}{P_{cr}}}\right) - 1\right]. \quad \left(\frac{L}{2} \sqrt{\frac{P_{cr}}{EI}} = \frac{\pi}{2}, \quad P_{cr} = \frac{\pi^2 EI}{L_e^2}\right)
\]
Eccentric Loading: The Secant Formula

- **Maximum stress**
  \[
  \sigma_{\text{max}} = \frac{P}{A} + \frac{M_{\text{max}} c}{I} = \frac{P}{A} + \frac{P\left(w_{\text{max}} + e\right)c}{I}
  \]
  \[
  = \frac{P}{A} \left[ 1 + \frac{ec}{i_r^2} \sec \left( \frac{\pi}{2} \sqrt{\frac{P}{P_{cr}}} \right) \right]
  \]
  \[
  = \frac{P}{A} \left[ 1 + \frac{ec}{i_r^2} \sec \left( \frac{L}{2} \sqrt{\frac{P}{EI}} \right) \right]
  \]
  \[
  = \frac{P}{A} \left[ 1 + \frac{ec}{i_r^2} \sec \left( \frac{1}{2} \frac{L}{i_r} \sqrt{\frac{P}{EA}} \right) \right]
  \]

- **Secant Formula**
  \[
  \frac{P}{A} = \sigma_{\text{max}} \left( 1 + \frac{ec}{i_r^2} \right)
  \]
  \[
  \frac{P}{A} = \sigma_{\text{max}} \left( 1 + \frac{ec}{i_r^2} \sec \left( \frac{1}{2} \frac{L}{i_r} \sqrt{\frac{P}{EA}} \right) \right)
  \]

- For small slenderness ratio: \( \frac{P}{A} \approx \sigma_{\text{max}} \left( 1 + \frac{ec}{i_r^2} \right) \).

- For large slenderness ratio, the curves get very close to Euler’s curve, and thus that the effect of the eccentricity of the loading becomes negligible.

- The secant formula is chiefly useful for intermediate values of slenderness ratio.
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