CHAPTER 18

Kinetics of Rigid Bodies in Three Dimensions
18.1 INTRODUCTION

In Chaps. 16 and 17 we were concerned with the plane motion of rigid bodies and of systems of rigid bodies. In Chap. 16 and in the second half of Chap. 17 (momentum method), our study was further restricted to that of plane slabs and of bodies symmetrical with respect to the reference plane. However, many of the fundamental results obtained in these two chapters remain valid in the case of the motion of a rigid body in three dimensions.

For example, the two fundamental equations

\[ \Sigma F = m\vec{a} \]  \hspace{1cm} (18.1)
\[ \Sigma M_G = \vec{H}_G \]  \hspace{1cm} (18.2)

on which the analysis of the plane motion of a rigid body was based, remain valid in the most general case of motion of a rigid body. As indicated in Sec. 16.2, these equations express that the system of the external forces is equipollent to the system consisting of the vector \( m\vec{a} \) attached at \( G \) and the couple of moment \( \vec{H}_G \) (Fig. 18.1). However, the relation \( \vec{H}_G = \vec{T}\omega \), which enabled us to determine the angular momentum of a rigid slab and which played an important part in the solution of problems involving the plane motion of slabs and bodies symmetrical with respect to the reference plane, ceases to be valid in the case of nonsymmetrical bodies or three-dimensional motion. Thus in the first part of the chapter, in Sec. 18.2, a more general method for computing the angular momentum \( \vec{H}_G \) of a rigid body in three dimensions will be developed.

Similarly, although the main feature of the impulse-momentum method discussed in Sec. 17.7, namely, the reduction of the momenta of the particles of a rigid body to a linear momentum vector \( m\vec{v} \) attached at the mass center \( G \) of the body and an angular momentum couple \( \vec{H}_G \), remains valid, the relation \( \vec{H}_G = \vec{T}\omega \) must be discarded and replaced by the more general relation developed in Sec. 18.2 before this method can be applied to the three-dimensional motion of a rigid body (Sec. 18.3).

We also note that the work-energy principle (Sec. 17.2) and the principle of conservation of energy (Sec. 17.6) still apply in the case
of the motion of a rigid body in three dimensions. However, the expression obtained in Sec. 17.4 for the kinetic energy of a rigid body in plane motion will be replaced by a new expression developed in Sec. 18.4 for a rigid body in three-dimensional motion.

In the second part of the chapter, you will first learn to determine the rate of change $H_G$ of the angular momentum $H_G$ of a three-dimensional rigid body, using a rotating frame of reference with respect to which the moments and products of inertia of the body remain constant (Sec. 18.5). Equations (18.1) and (18.2) will then be expressed in the form of free-body-diagram equations, which can be used to solve various problems involving the three-dimensional motion of rigid bodies (Secs. 18.6 through 18.8).

The last part of the chapter (Secs. 18.9 through 18.11) is devoted to the study of the motion of the gyroscope or, more generally, of an axisymmetrical body with a fixed point located on its axis of symmetry. In Sec. 18.10, the particular case of the steady precession of a gyroscope will be considered, and, in Sec. 18.11, the motion of an axisymmetrical body subjected to no force, except its own weight, will be analyzed.

**18.2 Angular Momentum of a Rigid Body in Three Dimensions**

In this section you will see how the angular momentum $H_G$ of a body about its mass center $G$ can be determined from the angular velocity $\omega$ of the body in the case of three-dimensional motion.

According to Eq. (14.24), the angular momentum of the body about $G$ can be expressed as

$$H_G = \sum_{i=1}^{n} (r_i' \times v_i' \Delta m_i) \quad (18.3)$$

where $r_i'$ and $v_i'$ denote, respectively, the position vector and the velocity of the particle $P_i$ of mass $\Delta m_i$, relative to the centroidal frame $Gxyz$ (Fig. 18.2). But $v_i' = \omega \times r_i'$, where $\omega$ is the angular
velocity of the body at the instant considered. Substituting into (18.3), we have

\[ \mathbf{H}_G = \sum_{i=1}^{n} \left[ \mathbf{r}'_i \times (\mathbf{\omega} \times \mathbf{r}'_i) \right] \Delta m_i \]

Recalling the rule for determining the rectangular components of a vector product (Sec. 3.5), we obtain the following expression for the \( x \) component of the angular momentum:

\[ H_x = \sum_{i=1}^{n} \left[ y_i (\mathbf{\omega} \times \mathbf{r}'_i)_z - z_i (\mathbf{\omega} \times \mathbf{r}'_i)_y \right] \Delta m_i \]

\[ = \sum_{i=1}^{n} \left[ y_i (\mathbf{\omega}_y x_i - \mathbf{\omega}_z x_i) - z_i (\mathbf{\omega}_x x_i - \mathbf{\omega}_y x_i) \right] \Delta m_i \]

\[ = \omega_x \sum_i (y_i^2 + z_i^2) \Delta m_i - \omega_y \sum_i x_i y_i \Delta m_i - \omega_z \sum_i z_i \Delta m_i \]

Replacing the sums by integrals in this expression and in the two similar expressions which are obtained for \( H_y \) and \( H_z \), we have

\[ H_x = \omega_x \int (y^2 + z^2) \, dm - \omega_y \int f_{xy} \, dm - \omega_z \int f_{zx} \, dm \]
\[ H_y = -\omega_x \int f_{xy} \, dm + \omega_y \int (z^2 + x^2) \, dm - \omega_z \int f_{yz} \, dm \]
\[ H_z = -\omega_x \int f_{zx} \, dm - \omega_y \int f_{yz} \, dm + \omega_z \int (x^2 + y^2) \, dm \] (18.4)

We note that the integrals containing squares represent the \textit{centroidal mass moments of inertia} of the body about the \( x \), \( y \), and \( z \) axes, respectively (Sec. 9.11); we have

\[ \bar{I}_x = \int (y^2 + z^2) \, dm \quad \bar{I}_y = \int (z^2 + x^2) \, dm \]
\[ \bar{I}_z = \int (x^2 + y^2) \, dm \] (18.5)

Similarly, the integrals containing products of coordinates represent the \textit{centroidal mass products of inertia} of the body (Sec. 9.16); we have

\[ \bar{I}_{xy} = \int xy \, dm \quad \bar{I}_{yz} = \int yz \, dm \quad \bar{I}_{zx} = \int zx \, dm \] (18.6)

Substituting from (18.5) and (18.6) into (18.4), we obtain the components of the angular momentum \( \mathbf{H}_G \) of the body about its mass center:

\[ H_x = +\bar{I}_x \mathbf{\omega}_x - \bar{I}_{xy} \mathbf{\omega}_y - \bar{I}_{xz} \mathbf{\omega}_z \]
\[ H_y = -\bar{I}_{yx} \mathbf{\omega}_x + \bar{I}_y \mathbf{\omega}_y - \bar{I}_{yz} \mathbf{\omega}_z \]
\[ H_z = -\bar{I}_{zx} \mathbf{\omega}_x - \bar{I}_{zy} \mathbf{\omega}_y + \bar{I}_z \mathbf{\omega}_z \] (18.7)
The relations (18.7) show that the operation which transforms the vector \( \mathbf{v} \) into the vector \( \mathbf{H}_G \) (Fig. 18.3) is characterized by the array of moments and products of inertia
\[
\begin{pmatrix}
I_x & -I_{xy} & -I_{xz} \\
-I_{yx} & I_y & -I_{yz} \\
-I_{zx} & -I_{zy} & I_z
\end{pmatrix}
\] (18.8)

The array (18.8) defines the inertia tensor of the body at its mass center \( G \).† A new array of moments and products of inertia would be obtained if a different system of axes were used. The transformation characterized by this new array, however, would still be the same. Clearly, the angular momentum \( \mathbf{H}_G \) corresponding to a given angular velocity \( \mathbf{v} \) is independent of the choice of the coordinate axes. As was shown in Secs. 9.17 and 9.18, it is always possible to select a system of axes \( Gx'y'z' \), called principal axes of inertia, with respect to which all the products of inertia of a given body are zero. The array (18.8) takes then the diagonalized form
\[
\begin{pmatrix}
I_x' & 0 & 0 \\
0 & I_y' & 0 \\
0 & 0 & I_z'
\end{pmatrix}
\] (18.9)

where \( I_x' \), \( I_y' \), \( I_z' \) represent the principal centroidal moments of inertia of the body, and the relations (18.7) reduce to
\[
\begin{align*}
H_x' &= I_x' \omega_x' \\
H_y' &= I_y' \omega_y' \\
H_z' &= I_z' \omega_z'
\end{align*}
\] (18.10)

We note that if the three principal centroidal moments of inertia \( I_x', I_y', I_z' \) are equal, the components \( H_x', H_y', H_z' \) of the angular momentum about \( G \) are proportional to the components \( \omega_x', \omega_y', \omega_z' \) of the angular velocity, and the vectors \( \mathbf{H}_G \) and \( \mathbf{v} \) are collinear. In general, however, the principal moments of inertia will be different, and the vectors \( \mathbf{H}_G \) and \( \mathbf{v} \) will have different directions, except when two of the three components of \( \omega \) happen to be zero, i.e., when \( \omega \) is directed along one of the coordinate axes. Thus, the angular momentum \( \mathbf{H}_G \) of a rigid body and its angular velocity \( \mathbf{v} \) have the same direction if, and only if, \( \omega \) is directed along a principal axis of inertia.‡

†Setting \( I_1 = I_{11}, I_2 = I_{22}, I_3 = I_{33} \) and \( -I_{12} = I_{12}, -I_{13} = I_{13}, \) etc., we may write the inertia tensor (18.8) in the standard form
\[
\begin{pmatrix}
I_{11} & I_{12} & I_{13} \\
I_{21} & I_{22} & I_{23} \\
I_{31} & I_{32} & I_{33}
\end{pmatrix}
\]

Denoting by \( H_x, H_y, H_z \) the components of the angular momentum \( \mathbf{H}_G \) and by \( \omega_x, \omega_y, \omega_z \) the components of the angular velocity \( \omega \), we can write the relations (18.7) in the form
\[
H_i = \sum_j I_{ij} \omega_j
\]

where \( i \) and \( j \) take the values 1, 2, 3. The quantities \( I_{ij} \) are said to be the components of the inertia tensor. Since \( I_{ij} = I_{ji} \), the inertia tensor is a symmetric tensor of the second order.

‡In the particular case when \( I_x = I_y = I_z \), any line through \( G \) can be considered as a principal axis of inertia, and the vectors \( \mathbf{H}_G \) and \( \mathbf{v} \) are always collinear.
Since this condition is satisfied in the case of the plane motion of a rigid body symmetrical with respect to the reference plane, we were able in Secs. 16.3 and 17.8 to represent the angular momentum $H_G$ of such a body by the vector $I\omega$. We must realize, however, that this result cannot be extended to the case of the plane motion of a non-symmetrical body, or to the case of the three-dimensional motion of a rigid body. Except when $\omega$ happens to be directed along a principal axis of inertia, the angular momentum and angular velocity of a rigid body have different directions, and the relation (18.7) or (18.10) must be used to determine $H_G$ from $\omega$.

**Reduction of the Momenta of the Particles of a Rigid Body to a Momentum Vector and a Couple at $G$.** We saw in Sec. 17.8 that the system formed by the momenta of the various particles of a rigid body can be reduced to a vector $L$ attached at the mass center $G$ of the body, representing the linear momentum of the body, and to a couple $H_G$, representing the angular momentum of the body about $G$ (Fig. 18.4). We are now in a position to determine the vector $L$ and the couple $H_G$ in the most general case of three-dimensional motion of a rigid body. As in the case of the two-dimensional motion considered in Sec. 17.8, the linear momentum $L$ of the body is equal to the product $m\vec{v}$ of its mass $m$ and the velocity $\vec{v}$ of its mass center $G$. The angular momentum $H_G$, however, can no longer be obtained by simply multiplying the angular velocity $\omega$ of the body by the scalar $I$; it must now be obtained from the components of $\omega$ and from the centroidal moments and products of inertia of the body through the use of Eq. (18.7) or (18.10).

We should also note that once the linear momentum $m\vec{v}$ and the angular momentum $H_G$ of a rigid body have been determined, its angular momentum $H_O$ about any given point $O$ can be obtained by adding the moments about $O$ of the vector $m\vec{v}$ and of the couple $H_G$. We write

$$H_O = \vec{r} \times m\vec{v} + H_G \quad (18.11)$$
Angular Momentum of a Rigid Body Constrained to Rotate about a Fixed Point. In the particular case of a rigid body constrained to rotate in three-dimensional space about a fixed point \(O\) (Fig. 18.5a), it is sometimes convenient to determine the angular momentum \(H_O\) of the body about the fixed point \(O\). While \(H_O\) could be obtained by first computing \(H_G\) as indicated above and then using Eq. (18.11), it is often advantageous to determine \(H_O\) directly from the angular velocity \(\omega\) of the body and its moments and products of inertia with respect to a frame \(Oxyz\) centered at the fixed point \(O\). Recalling Eq. (14.7), we write

\[
H_O = \sum_{i=1}^{n} (r_i \times v_i \Delta m_i)
\]

where \(r_i\) and \(v_i\) denote, respectively, the position vector and the velocity of the particle \(P_i\) with respect to the fixed frame \(Oxyz\). Substituting \(v_i = \omega \times r_i\), and after manipulations similar to those used in the earlier part of this section, we find that the components of the angular momentum \(H_O\) (Fig. 18.5b) are given by the relations

\[
\begin{align*}
H_x &= +I_x \omega_x - I_{yx} \omega_y - I_{xz} \omega_z \\
H_y &= -I_{yx} \omega_x + I_y \omega_y - I_{yz} \omega_z \\
H_z &= -I_{xz} \omega_x - I_{yz} \omega_y + I_z \omega_z
\end{align*}
\]  

(18.13)

where the moments of inertia \(I_x, I_y, I_z\) and the products of inertia \(I_{yx}, I_{yz}, I_{zx}\) are computed with respect to the frame \(Oxyz\) centered at the fixed point \(O\).

18.3 Application of the Principle of Impulse and Momentum to the Three-Dimensional Motion of a Rigid Body

Before we can apply the fundamental equation (18.2) to the solution of problems involving the three-dimensional motion of a rigid body, we must learn to compute the derivative of the vector \(H_G\). This will be done in Sec. 18.5. The results obtained in the preceding section can, however, be used right away to solve problems by the impulse-momentum method.

Recalling that the system formed by the momenta of the particles of a rigid body reduces to a linear momentum vector \(m\vec{V}\)
attached at the mass center $G$ of the body and an angular momentum couple $H_G$: we represent graphically the fundamental relation

\[ \text{Syst Momenta}_1 + \text{Syst Ext Imp}_{1\rightarrow 2} = \text{Syst Momenta}_2 \]  

(17.4)

by means of the three sketches shown in Fig. 18.6. To solve a given problem, we can use these sketches to write appropriate component and moment equations, keeping in mind that the components of the angular momentum $H_G$ are related to the components of the angular velocity $\omega$ by Eqs. (18.7) of the preceding section.

In solving problems dealing with the motion of a body rotating about a fixed point $O$, it will be convenient to eliminate the impulse of the reaction at $O$ by writing an equation involving the moments of the momenta and impulses about $O$. We recall that the angular momentum $H_O$ of the body about the fixed point $O$ can be obtained either directly from Eqs. (18.13) or by first computing its linear momentum $m\vec{v}$ and its angular momentum $H_G$ and then using Eq. (18.11).

*18.4 KINETIC ENERGY OF A RIGID BODY IN THREE DIMENSIONS

Consider a rigid body of mass $m$ in three-dimensional motion. We recall from Sec. 14.6 that if the absolute velocity $\vec{v}_i$ of each particle $P_i$ of the body is expressed as the sum of the velocity $\vec{v}$ of the mass center $G$ of the body and of the velocity $\vec{v}'_i$ of the particle relative to a frame $Gxyz$ attached to $G$ and of fixed orientation (Fig. 18.7), the kinetic energy of the system of particles forming the rigid body can be written in the form

\[ T = \frac{1}{2} m\vec{v}^2 + \frac{1}{2} \sum_{i=1}^{n} \Delta m_i \vec{v}'_i^2 \]  

(18.14)

where the last term represents the kinetic energy $T'$ of the body relative to the centroidal frame $Gxyz$. Since $\vec{v}'_i = |\vec{v}'_i| = |\omega \times \vec{r}'_i|$, we write

\[ T' = \frac{1}{2} \sum_{i=1}^{n} \Delta m_i \vec{v}'_i^2 = \frac{1}{2} \sum_{i=1}^{n} |\omega \times \vec{r}'_i|^2 \Delta m_i \]
Expressing the square in terms of the rectangular components of the vector product, and replacing the sums by integrals, we have

\[
T' = \frac{1}{2} \int \left[ (\omega_x y - \omega_y x)^2 + (\omega_y z - \omega_z y)^2 + (\omega_z x - \omega_x z)^2 \right] \, dm
\]

or, recalling the relations (18.5) and (18.6),

\[
T' = \frac{1}{2} \left( I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 - 2I_{xy}\omega_y\omega_x - 2I_{yz}\omega_z\omega_y - 2I_{zx}\omega_x\omega_z \right)
\]

Substituting into (18.14) the expression (18.15) we have just obtained for the kinetic energy of the body relative to centroidal axes, we write

\[
T = \frac{1}{2} m \bar{v}^2 + \frac{1}{2} \left( I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 - 2I_{xy}\omega_y\omega_x - 2I_{yz}\omega_z\omega_y - 2I_{zx}\omega_x\omega_z \right)
\]

If the axes of coordinates are chosen so that they coincide at the instant considered with the principal axes \(x', \ y', \ z'\) of the body, the relation obtained reduces to

\[
T = \frac{1}{2} m \bar{v}^2 + \frac{1}{2} \left( I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 \right)
\]

where \(\bar{v}\) = velocity of mass center
\(\omega\) = angular velocity
\(m\) = mass of rigid body
\(I_x, I_y, I_z\) = principal centroidal moments of inertia

The results we have obtained enable us to apply to the three-dimensional motion of a rigid body the principles of work and energy (Sec. 17.2) and conservation of energy (Sec. 17.6).

**Kinetic Energy of a Rigid Body with a Fixed Point.** In the particular case of a rigid body rotating in three-dimensional space about a fixed point \(O\), the kinetic energy of the body can be expressed in terms of its moments and products of inertia with respect to axes attached at \(O\) (Fig. 18.8). Recalling the definition of the kinetic energy of a system of particles, and substituting \(v_i = |v_i| = |\omega \times r_i|\), we write

\[
T = \frac{1}{2} \sum_{i=1}^{n} \Delta m_i v_i^2 = \frac{1}{2} \sum_{i=1}^{n} |\omega \times r_i|^2 \Delta m_i
\]

Manipulations similar to those used to derive Eq. (18.15) yield

\[
T = \frac{1}{2} \left( I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 - 2I_{xy}\omega_y\omega_x - 2I_{yz}\omega_z\omega_y - 2I_{zx}\omega_x\omega_z \right)
\]

or, if the principal axes \(x', \ y', \ z'\) of the body at the origin \(O\) are chosen as coordinate axes,

\[
T = \frac{1}{2} (I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2)
\]
SAMPLE PROBLEM 18.1

A rectangular plate of mass \( m \) suspended from two wires at \( A \) and \( B \) is hit at \( D \) in a direction perpendicular to the plate. Denoting by \( \mathbf{F} \Delta t \) the impulse applied at \( D \), determine immediately after the impact (a) the velocity of the mass center \( G \), (b) the angular velocity of the plate.

SOLUTION

Assuming that the wires remain taut and thus that the components \( \vec{v}_y \) of \( \mathbf{v} \) and \( \omega_z \) of \( \mathbf{\omega} \) are zero after the impact, we have

\[
\mathbf{v} = \vec{v}_x \mathbf{i} + \vec{v}_z \mathbf{k} \quad \mathbf{\omega} = \omega_x \mathbf{i} + \omega_y \mathbf{j}
\]

and since the \( x, y, z \) axes are principal axes of inertia,

\[
\mathbf{H}_G = \mathbf{I}_x \omega_x \mathbf{i} + \mathbf{I}_y \omega_y \mathbf{j} \quad \mathbf{H}_G = \frac{1}{12} mb^2 \omega_x \mathbf{i} + \frac{1}{12} ma^2 \omega_y \mathbf{j}
\]  

(1)

Principle of Impulse and Momentum. Since the initial momenta are zero, the system of the impulses must be equivalent to the system of the final momenta:

\[
\begin{align*}
\mathbf{T}_A \Delta t & = \mathbf{T}_B \Delta t = \mathbf{W} \Delta t \\
\mathbf{F} \Delta t & = \mathbf{H}_y \mathbf{i} + \mathbf{H}_z \mathbf{j} \\
\end{align*}
\]

(2)

a. Velocity of Mass Center. Equating the components of the impulses and momenta in the \( x \) and \( z \) directions:

\[
\begin{align*}
x \text{ components:} & \quad 0 = m \vec{v}_x \quad \vec{v}_z = 0 \\
z \text{ components:} & \quad -F \Delta t = m \vec{v}_z \\ & \quad \vec{v}_x = -F \Delta t/m \\ & \quad \mathbf{v} = \vec{v}_x \mathbf{i} + \vec{v}_z \mathbf{k} \\ & \quad \mathbf{v} = -(F \Delta t/m) \mathbf{k}
\end{align*}
\]

b. Angular Velocity. Equating the moments of the impulses and momenta about the \( x \) and \( y \) axes:

About \( x \) axis:

\[
\frac{1}{2} b F \Delta t = H_x
\]

About \( y \) axis:

\[
-\frac{1}{2} a F \Delta t = H_y
\]

\[
\mathbf{H}_G = H_x \mathbf{i} + H_y \mathbf{j} \quad \mathbf{H}_G = \frac{1}{2} b F \Delta t \mathbf{i} - \frac{1}{2} a F \Delta t \mathbf{j}
\]

Comparing Eqs. (1) and (2), we conclude that

\[
\begin{align*}
\omega_x & = \frac{6 F \Delta t/m b}{6 F \Delta t/m a} \quad \omega_y = -\frac{6 F \Delta t/m a}{6 F \Delta t/m b} \\
\mathbf{\omega} & = \omega_x \mathbf{i} + \omega_y \mathbf{j} \\
\mathbf{\omega} & = \frac{(6 F \Delta t/m b)(a \mathbf{i} - b \mathbf{j})}{6 F \Delta t/m b}
\end{align*}
\]

We note that \( \mathbf{\omega} \) is directed along the diagonal \( AC \).

Remark: Equating the \( y \) components of the impulses and momenta, and their moments about the \( z \) axis, we obtain two additional equations which yield \( T_A = T_B = \frac{1}{2} W \). We thus verify that the wires remain taut and that our assumption was correct.
A homogeneous disk of radius \( r \) and mass \( m \) is mounted on an axle \( OG \) of length \( L \) and negligible mass. The axle is pivoted at the fixed point \( O \), and the disk is constrained to roll on a horizontal floor. Knowing that the disk rotates counterclockwise at the rate \( \omega_1 \) about the axle \( OG \), determine (a) the angular velocity of the disk, (b) its angular momentum about \( O \), (c) its kinetic energy, (d) the vector and couple at \( G \) equivalent to the momenta of the particles of the disk.

**SOLUTION**

**a. Angular Velocity.** As the disk rotates about the axle \( OG \) it also rotates with the axle about the \( y \) axis at a rate \( \omega_2 \) clockwise. The total angular velocity of the disk is therefore

\[
\omega = \omega_1 \hat{i} - \omega_2 \hat{j}
\]  

(1)

To determine \( \omega_2 \) we write that the velocity of \( C \) is zero:

\[
v_C = \omega \times r_C = 0
\]

\[
(v_1 \hat{i} - v_2 \hat{j}) \times (LI - r) = 0
\]

\[
(L\omega_2 - r\omega_1)\hat{k} = 0 \quad \omega_2 = r\omega_1/L
\]

Substituting into (1) for \( \omega_2 \):

\[
\omega = \omega_1 \hat{i} - (r\omega_1/L) \hat{j}
\]

**b. Angular Momentum about \( O \).** Assuming the axle to be part of the disk, we can consider the disk to have a fixed point at \( O \). Since the \( x, y, z \) axes are principal axes of inertia for the disk,

\[
H_x = I_x \omega_x = \frac{1}{2}mr^2 \omega_1
\]

\[
H_y = I_y \omega_y = (mL^2 + \frac{1}{2}mr^2)(-r\omega_1/L)
\]

\[
H_z = I_z \omega_z = (mL^2 + \frac{1}{2}mr^2)0 = 0
\]

\[
H_y = \frac{1}{2}mr^2 \omega_1 - m(L^2 + \frac{1}{4}r^2)(r\omega_1/L) \hat{j}
\]

**c. Kinetic Energy.** Using the values obtained for the moments of inertia and the components of \( \omega \), we have

\[
T = \frac{1}{2}(I_x \omega_x^2 + I_y \omega_y^2 + I_z \omega_z^2) = \frac{1}{2}\left[\frac{1}{2}mr^2 \omega_1^2 + m(L^2 + \frac{1}{4}r^2)(-r\omega_1/L)^2\right]
\]

\[
T = \frac{1}{2}mr^2 \left(6 + \frac{r^2}{L^2}\right) \omega_1^2
\]

**d. Momentum Vector and Couple at \( G \).** The linear momentum vector \( m\vec{v} \) and the angular momentum couple \( \vec{H}_G \) are

\[
m\vec{v} = m\omega_1 \hat{k}
\]

and

\[
\vec{H}_G = \vec{I}_x \omega_x \hat{i} + \vec{I}_y \omega_y \hat{j} + \vec{I}_z \omega_z \hat{k} = \frac{1}{2}mr^2 \omega_1 \hat{i} + \frac{1}{2}mr^2(-r\omega_1/L) \hat{j}
\]

\[
\vec{H}_G = \frac{1}{2}mr^2 \omega_1 \left(\hat{i} - \frac{r}{2L} \hat{j}\right)
\]
In this lesson you learned to compute the angular momentum of a rigid body in three dimensions and to apply the principle of impulse and momentum to the three-dimensional motion of a rigid body. You also learned to compute the kinetic energy of a rigid body in three dimensions. It is important for you to keep in mind that, except for very special situations, the angular momentum of a rigid body in three dimensions cannot be expressed as the product $I \omega$ and, therefore, will not have the same direction as the angular velocity $\omega$ (Fig. 18.3).

1. To compute the angular momentum $H_G$ of a rigid body about its mass center $G$, you must first determine the angular velocity $\omega$ of the body with respect to a system of axes centered at $G$ and of fixed orientation. Since you will be asked in this lesson to determine the angular momentum of the body at a given instant only, select the system of axes which will be most convenient for your computations.

   a. If the principal axes of inertia of the body at $G$ are known, use these axes as coordinate axes $x'$, $y'$, and $z'$, since the corresponding products of inertia of the body will be equal to zero. Resolve $\omega$ into components $\omega_{x'}$, $\omega_{y'}$, and $\omega_{z'}$ along these axes and compute the principal moments of inertia $I_{x'}$, $I_{y'}$, and $I_{z'}$. The corresponding components of the angular momentum $H_G$ are

   $$H_{x'} = I_{x'} \omega_{x'} \quad H_{y'} = I_{y'} \omega_{y'} \quad H_{z'} = I_{z'} \omega_{z'} \quad (18.10)$$

   b. If the principal axes of inertia of the body at $G$ are not known, you must use Eqs. (18.7) to determine the components of the angular momentum $H_G$. These equations require prior computation of the products of inertia of the body as well as prior computation of its moments of inertia with respect to the selected axes.

   c. The magnitude and direction cosines of $H_G$ are obtained from formulas similar to those used in Statics [Sec. 2.12]. We have

   $$H_G = \sqrt{H_x^2 + H_y^2 + H_z^2}$$

   $$\cos \theta_x = \frac{H_x}{H_G} \quad \cos \theta_y = \frac{H_y}{H_G} \quad \cos \theta_z = \frac{H_z}{H_G}$$

   d. Once $H_G$ has been determined, you can obtain the angular momentum of the body about any given point $O$ by observing from Fig. (18.4) that

   $$H_O = \vec{r} \times m\vec{v} + H_G \quad (18.11)$$

   where $\vec{r}$ is the position vector of $G$ relative to $O$, and $m\vec{v}$ is the linear momentum of the body.

2. To compute the angular momentum $H_O$ of a rigid body with a fixed point $O$, follow the procedure described in paragraph 1, except that you should now use axes centered at the fixed point $O$.

   a. If the principal axes of inertia of the body at $O$ are known, resolve $\omega$ into components along these axes [Sample Prob. 18.2]. The corresponding components of the angular momentum $H_O$ are obtained from equations similar to Eqs. (18.10).
b. If the principal axes of inertia of the body at \( O \) are not known, you must compute the products as well as the moments of inertia of the body with respect to the axes that you have selected and use Eqs. (18.13) to determine the components of the angular momentum \( \mathbf{H}_O \).

3. To apply the principle of impulse and momentum to the solution of a problem involving the three-dimensional motion of a rigid body, you will use the same vector equation that you used for plane motion in Chap. 17,

\[
\text{Syst Momenta}_1 + \text{Syst Ext Imp}_{1\rightarrow 2} = \text{Syst Momenta}_2 \quad (17.4)
\]

where the initial and final systems of momenta are each represented by a linear-momentum vector \( m\mathbf{v} \) and an angular-momentum couple \( \mathbf{H}_C \). Now, however, these vector-and-couple systems should be represented in three dimensions as shown in Fig. 18.6, and \( \mathbf{H}_C \) should be determined as explained in paragraph 1.

a. In problems involving the application of a known impulse to a rigid body, draw the free-body-diagram equation corresponding to Eq. (17.4). Equating the components of the vectors involved, you will determine the final linear momentum \( m\mathbf{v} \) of the body and, thus, the corresponding velocity \( \mathbf{v} \) of its mass center. Equating moments about \( G \), you will determine the final angular momentum \( \mathbf{H}_C \) of the body. You will then substitute the values obtained for the components of \( \mathbf{H}_C \) into Eqs. (18.10) or (18.7) and solve these equations for the corresponding values of the components of the angular velocity \( \mathbf{\omega} \) of the body [Sample Prob. 18.1].

b. In problems involving unknown impulses, draw the free-body-diagram equation corresponding to Eq. (17.4) and write equations which do not involve the unknown impulses. Such equations can be obtained by equating moments about the point or line of impact.

4. To compute the kinetic energy of a rigid body with a fixed point \( O \), resolve the angular velocity \( \mathbf{\omega} \) into components along axes of your choice and compute the moments and products of inertia of the body with respect to these axes. As was the case for the computation of the angular momentum, use the principal axes of inertia \( x', y', \) and \( z' \) if you can easily determine them. The products of inertia will then be zero [Sample Prob. 18.2], and the expression for the kinetic energy will reduce to

\[
T = \frac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2) \quad (18.20)
\]

If you must use axes other than the principal axes of inertia, the kinetic energy of the body should be expressed as shown in Eq. (18.19).

5. To compute the kinetic energy of a rigid body in general motion, consider the motion as the sum of a translation with the mass center \( G \) and a rotation about \( G \). The kinetic energy associated with the translation is \( \frac{1}{2}m\mathbf{v}^2 \). If principal axes of inertia can be used, the kinetic energy associated with the rotation about \( G \) can be expressed in the form used in Eq. (18.20). The total kinetic energy of the rigid body is then

\[
T = \frac{1}{2}m\mathbf{v}^2 + \frac{1}{2}(\bar{I}_x\omega_x^2 + \bar{I}_y\omega_y^2 + \bar{I}_z\omega_z^2) \quad (18.17)
\]

If you must use axes other than the principal axes of inertia to determine the kinetic energy associated with the rotation about \( G \), the total kinetic energy of the body should be expressed as shown in Eq. (18.16).
**PROBLEMS**

18.1 Two uniform rods $AB$ and $CE$, each of mass 1.5 kg and length 600 mm, are welded to each other at their midpoints. Knowing that this assembly has an angular velocity of constant magnitude $\omega = 12 \text{ rad/s}$, determine the magnitude and direction of the angular momentum $\mathbf{H}_D$ of the assembly about $D$.

![Fig. P18.1](image1)

18.2 A thin, homogeneous disk of mass $m$ and radius $r$ spins at the constant rate $\omega_1$ about an axle held by a fork-ended vertical rod which rotates at the constant rate $\omega_2$. Determine the angular momentum $\mathbf{H}_G$ of the disk about its mass center $G$.

![Fig. P18.2](image2)

18.3 A thin homogeneous square of mass $m$ and side $a$ is welded to a vertical shaft $AB$ with which it forms an angle of $45^\circ$. Knowing that the shaft rotates with an angular velocity $\omega$, determine the angular momentum of the plate about $A$.

![Fig. P18.3](image3)

18.4 A homogeneous disk of mass $m$ and radius $r$ is mounted on the vertical shaft $AB$. The normal to the disk at $G$ forms an angle $\beta = 25^\circ$ with the shaft. Knowing that the shaft has a constant angular velocity $\omega$, determine the angle $\theta$ formed by the shaft $AB$ and the angular momentum $\mathbf{H}_G$ of the disk about its mass center $G$.

![Fig. P18.4](image4)
18.5 A thin disk of weight \( W = 10 \) lb rotates at the constant rate \( \omega_2 = 15 \text{ rad/s} \) with respect to arm ABC, which itself rotates at the constant rate \( \omega_1 = 5 \text{ rad/s} \) about the \( y \) axis. Determine the angular momentum of the disk about its center \( C \).

![Fig. P18.5](image)

18.6 A homogeneous disk of weight \( W = 6 \) lb rotates at the constant rate \( \omega_1 = 16 \text{ rad/s} \) with respect to arm ABC, which is welded to a shaft \( DCE \) rotating at the constant rate \( \omega_2 = 8 \text{ rad/s} \). Determine the angular momentum \( \mathbf{H}_A \) of the disk about its center \( A \).

![Fig. P18.6](image)

18.7 A solid rectangular parallelepiped of mass \( m \) has a square base of side \( a \) and a length \( 2a \). Knowing that it rotates at the constant rate \( \omega \) about its diagonal \( AC' \) and that its rotation is observed from \( A \) as counterclockwise, determine (a) the magnitude of the angular momentum \( \mathbf{H}_G \) of the parallelepiped about its mass center \( G \), (b) the angle that \( \mathbf{H}_G \) forms with the diagonal \( AC' \).

![Fig. P18.7](image)

18.8 Solve Prob. 18.7, assuming that the solid rectangular parallelepiped has been replaced by a hollow one consisting of six thin metal plates welded together.

18.9 Determine the angular momentum of the disk of Prob. 18.5 about point \( A \).

18.10 Determine the angular momentum \( \mathbf{H}_D \) of the disk of Prob. 18.6 about point \( D \).
Kinetics of Rigid Bodies in Three Dimensions

18.11 The 30-kg projectile shown has a radius of gyration of 60 mm about its axis of symmetry \( G_x \) and a radius of gyration of 250 mm about the transverse axis \( G_y \). Its angular velocity \( \omega \) can be resolved into two components; one component, directed along \( G_x \), measures the rate of spin of the projectile, while the other component, directed along \( GD \), measures its rate of precession. Knowing that \( \theta = 5^\circ \) and that the angular momentum of the projectile about its mass center \( G \) is \( \mathbf{H}_G = (320 \text{ g} \cdot \text{m}^2/\text{s})\mathbf{i} - (9 \text{ g} \cdot \text{m}^2/\text{s})\mathbf{j} \), determine \( (a) \) the rate of spin, \( (b) \) the rate of precession.

18.12 Determine the angular momentum \( \mathbf{H}_A \) of the projectile of Prob. 18.11 about the center \( A \) of its base, knowing that its mass center \( G \) has a velocity \( \mathbf{v} \) of 650 m/s. Give your answer in terms of components respectively parallel to the \( x \) and \( y \) axes shown and to a third axis \( z \) pointing toward you.

18.13 \( (a) \) Show that the angular momentum \( \mathbf{H}_B \) of a rigid body about point \( B \) can be obtained by adding to the angular momentum \( \mathbf{H}_A \) of that body about point \( A \) the vector product of the vector \( \mathbf{r}_{A/B} \) drawn from \( B \) to \( A \) and the linear momentum \( m\mathbf{v} \) of the body:

\[
\mathbf{H}_B = \mathbf{H}_A + \mathbf{r}_{A/B} \times m\mathbf{v}
\]

\( (b) \) Further show that when a rigid body rotates about a fixed axis, its angular momentum is the same about any two points \( A \) and \( B \) located on the fixed axis \( (\mathbf{H}_A = \mathbf{H}_B) \) if, and only if, the mass center \( G \) of the body is located on the fixed axis.

18.14 Determine the angular momentum \( \mathbf{H}_O \) of the disk of Sample Prob. 18.2 from the expressions obtained for its linear momentum \( m\mathbf{v} \) and its angular momentum \( \mathbf{H}_G \), using Eqs. (18.11). Verify that the result obtained is the same as that obtained by direct computation.

18.15 A rod of uniform cross section is used to form the shaft shown. Denoting by \( m \) the total mass of the shaft and knowing that the shaft rotates with a constant angular velocity \( \omega \), determine \( (a) \) the angular momentum \( \mathbf{H}_C \) of the shaft about its mass center \( G \), \( (b) \) the angle formed by \( \mathbf{H}_G \) and the axis \( AB \).

18.16 The triangular plate shown has a mass of 7.5 kg and is welded to a vertical shaft \( AB \). Knowing that the plate rotates at the constant rate \( \omega = 12 \text{ rad/s} \), determine its angular momentum about \( (a) \) point \( C \), \( (b) \) point \( A \). (Hint: To solve part \( b \) find \( \mathbf{v} \) and use the property indicated in part \( a \) of Prob. 18.13.)

18.17 The triangular plate shown has a mass of 7.5 kg and is welded to a vertical shaft \( AB \). Knowing that the plate rotates at the constant rate \( \omega = 12 \text{ rad/s} \), determine its angular momentum about \( (a) \) point \( C \), \( (b) \) point \( B \). (See hint of Prob. 18.16.)
18.18 Determine the angular momentum of the shaft of Prob. 18.15 about (a) point A, (b) point B.

18.19 Two L-shaped arms, each weighing 4 lb, are welded at the third points of the 2-ft shaft AB. Knowing that shaft AB rotates at the constant rate \( \omega = 240 \text{ rpm} \), determine (a) the angular momentum of the body about A, (b) the angle formed by the angular momentum and shaft AB.

18.20 For the body of Prob. 18.19, determine (a) the angular momentum about B, (b) the angle formed by the angular momentum about shaft BA.

18.21 One of the sculptures displayed on a university campus consists of a hollow cube made of six aluminum sheets, each 5 × 5 ft, welded together and reinforced with internal braces of negligible weight. The cube is mounted on a fixed base at A and can rotate freely about its vertical diagonal AB. As she passes by this display on the way to a class in mechanics, an engineering student grabs corner C of the cube and pushes it for 1.2 s in a direction perpendicular to the plane ABC with an average force of 12.5 lb. Having observed that it takes 5 s for the cube to complete one full revolution, she flips out her calculator and proceeds to determine the weight of the cube. What is the result of her calculation? (Hint: The perpendicular distance from the diagonal joining two vertices of a cube to any of its other six vertices can be obtained by multiplying the side of the cube by \( \frac{1}{2} \sqrt{3} \).)

18.22 If the aluminum cube of Prob. 18.21 were replaced by a cube of the same size, made of six plywood sheets weighing 20 lb each, how long would it take for that cube to complete one full revolution if the student pushed its corner C in the same way that she pushed the corner of the aluminum cube?
18.23 Two circular plates, each of mass 4 kg, are rigidly connected by a rod \( AB \) of negligible mass and are suspended from point \( A \) as shown. Knowing that an impulse \( \mathbf{F} \Delta t = -2 \text{(2.4 N} \cdot \text{s)} \mathbf{k} \) is applied at point \( D \), determine (a) the velocity of the mass center \( G \) of the assembly, (b) the angular velocity of the assembly.

18.24 Two circular plates, each of mass 4 kg, are rigidly connected by a rod \( AB \) of negligible mass and are suspended from point \( A \) as shown. Knowing that an impulse \( \mathbf{F} \Delta t = (2.4 \text{ N} \cdot \text{s}) \mathbf{j} \) is applied at point \( D \), determine (a) the velocity of the mass center \( G \) of the assembly, (b) the angular velocity of the assembly.

18.25 A uniform rod of mass \( m \) is bent into the shape shown and is suspended from a wire attached at its mass center \( G \). The bent rod is hit at \( A \) in a direction perpendicular to the plane containing the rod (in the positive \( x \) direction). Denoting the corresponding impulse by \( \mathbf{F} \Delta t \), determine immediately after the impact (a) the velocity of the mass center \( G \), (b) the angular velocity of the rod.

18.26 Solve Prob. 18.25, assuming that the bent rod is hit at \( B \).

18.27 Three slender rods, each of mass \( m \) and length \( 2a \), are welded together to form the assembly shown. The assembly is hit at \( A \) in a vertical downward direction. Denoting the corresponding impulse by \( \mathbf{F} \Delta t \), determine immediately after the impact (a) the velocity of the mass center \( G \), (b) the angular velocity of the rod.

18.28 Solve Prob. 18.27, assuming that the assembly is hit at \( B \) in a direction opposite to that of the \( x \) axis.

18.29 A square plate of side \( a \) and mass \( m \) supported by a ball-and-socket joint at \( A \) is rotating about the \( y \) axis with a constant angular velocity \( \omega = \omega_0 \mathbf{j} \) when an obstruction is suddenly introduced at \( B \) in the \( xy \) plane. Assuming the impact at \( B \) to be perfectly plastic \( (e = 0) \), determine immediately after the impact (a) the angular velocity of the plate, (b) the velocity of its mass center \( G \).

18.30 Determine the impulse exerted on the plate of Prob. 18.29 during the impact by (a) the obstruction at \( B \), (b) the support at \( A \).
18.31 A rectangular plate of mass \( m \) is falling with a velocity \( \vec{v}_0 \) and no angular velocity when its corner \( C \) strikes an obstruction. Assuming the impact to be perfectly plastic \( (\epsilon = 0) \), determine the angular velocity of the plate immediately after the impact.

18.32 For the plate of Prob. 18.31, determine (a) the velocity of its mass center \( G \) immediately after the impact, (b) the impulse exerted on the plate by the obstruction during the impact.

18.33 A 2500-kg probe in orbit about the moon is 2.4 m high and has octagonal bases of sides 1.2 m. The coordinate axes shown are the principal centroidal axes of inertia of the probe, and its radii of gyration are \( k_x = 0.98 \text{ m}, \) \( k_y = 1.06 \text{ m}, \) and \( k_z = 1.02 \text{ m}. \) The probe is equipped with a main 500-N thruster \( E \) and with four 20-N thrusters \( A, B, C, \) and \( D \) which can expel fuel in the positive \( y \) direction. The probe has an angular velocity \( \vec{\omega} = (0.040 \text{ rad/s})\hat{i} + (0.060 \text{ rad/s})\hat{k} \) when two of the 20-N thrusters are used to reduce the angular velocity to zero. Determine (a) which of the thrusters should be used, (b) the operating time of each of these thrusters, (c) for how long the main thruster \( E \) should be activated if the velocity of the mass center of the probe is to remain unchanged.

18.34 Solve Prob. 18.33, assuming that the angular velocity of the probe is \( \vec{\omega} = (0.060 \text{ rad/s})\hat{i} - (0.040 \text{ rad/s})\hat{k}. \)

18.35 The coordinate axes shown represent the principal centroidal axes of inertia of a 3000-lb space probe whose radii of gyration are \( k_x = 1.375 \text{ ft}, k_y = 1.425 \text{ ft}, \) and \( k_z = 1.250 \text{ ft}. \) The probe has no angular velocity when a 5-oz meteorite strikes one of its solar panels at point \( A \) with a velocity \( \vec{v}_0 = (2400 \text{ ft/s})\hat{i} - (3000 \text{ ft/s})\hat{j} + (3200 \text{ ft/s})\hat{k} \) relative to the probe. Knowing that the meteorite emerges on the other side of the panel with no change in the direction of its velocity, but with a speed reduced by 20 percent, determine the final angular velocity of the probe.

18.36 The coordinate axes shown represent the principal centroidal axes of inertia of a 3000-lb space probe whose radii of gyration are \( k_x = 1.375 \text{ ft}, k_y = 1.425 \text{ ft}, \) and \( k_z = 1.250 \text{ ft}. \) The probe has no angular velocity when a 5-oz meteorite strikes one of its solar panels at point \( A \) and emerges on the other side of the panel with no change in the direction of its velocity, but with a speed reduced by 25 percent. Knowing that the final angular velocity of the probe is \( \vec{\omega} = (0.05 \text{ rad/s})\hat{i} - (0.12 \text{ rad/s})\hat{j} + \omega_k \hat{k} \) and that the
18.37 Denoting, respectively, by \( \omega \), \( H_O \), and \( T \) the angular velocity, the angular momentum, and the kinetic energy of a rigid body with a fixed point \( O \), (a) prove that \( H_O \cdot \omega = 2T \); (b) show that the angle \( \theta \) between \( \omega \) and \( H_O \) will always be acute.

18.38 Show that the kinetic energy of a rigid body with a fixed point \( O \) can be expressed as \( T = \frac{1}{2} I_{OL} \omega^2 \) where \( \omega \) is the instantaneous angular velocity of the body and \( I_{OL} \) is its moment of inertia about the line of action \( OL \) of \( \omega \). Derive this expression (a) from Eqs. (9.46) and (18.19), (b) by considering \( T \) as the sum of the kinetic energies of particles \( P_i \) describing circles of radius \( r_i \) about line \( OL \).

18.39 Determine the kinetic energy of the assembly of Prob. 18.1.

18.40 Determine the kinetic energy of the disk of Prob. 18.2.

18.41 Determine the kinetic energy of the plate of Prob. 18.3.

18.42 Determine the kinetic energy of the disk of Prob. 18.4.

18.43 Determine the kinetic energy of the rod of Prob. 18.15.

18.44 Determine the kinetic energy of the triangular plate of Prob. 18.16.

18.45 Determine the kinetic energy of the body of Prob. 18.19.

18.46 Determine the kinetic energy imparted to the cube of Prob. 18.21.

18.47 Determine the kinetic energy of the disk of Prob. 18.5.

18.48 Determine the kinetic energy of the disk of Prob. 18.6.

18.49 Determine the kinetic energy of the solid parallelepiped of Prob. 18.7.

18.50 Determine the kinetic energy of the hollow parallelepiped of Prob. 18.8.

18.51 Determine the kinetic energy lost when the plate of Prob. 18.29 hits the obstruction at \( B \).

18.52 Determine the kinetic energy lost when corner \( C \) of the plate of Prob. 18.31 hits the obstruction.

18.53 Determine the kinetic energy of the space probe of Prob. 18.35 in its motion about its mass center after its collision with the meteorite.

18.54 Determine the kinetic energy of the space probe of Prob. 18.36 in its motion about its mass center after its collision with the meteorite.
**18.5 MOTION OF A RIGID BODY IN THREE DIMENSIONS**

As indicated in Sec. 18.2, the fundamental equations

\[ \sum F = m \ddot{a} \]  \hspace{1cm} (18.1)

\[ \sum M_G = \dot{H}_G \]  \hspace{1cm} (18.2)

remain valid in the most general case of the motion of a rigid body. Before Eq. (18.2) could be applied to the three-dimensional motion of a rigid body, however, it was necessary to derive Eqs. (18.7), which relate the components of the angular momentum \( H_G \) and those of the angular velocity \( \omega \). It still remains for us to find an effective and convenient way for computing the components of the derivative \( \dot{H}_G \).

Since \( H_G \) represents the angular momentum of the body in its motion relative to centroidal axes \( GX'Y'Z' \) of fixed orientation (Fig. 18.9), and since \( \dot{H}_G \) represents the rate of change of \( H_G \) with respect to the same axes, it would seem natural to use components of \( H_G \) along the axes \( x', y', z' \) in writing the relations (18.7). But since the body rotates, its moments and products of inertia would change continually, and it would be necessary to determine their values as functions of the time. It is therefore more convenient to use axes \( x, y, z \) attached to the body, ensuring that its moments and products of inertia will maintain the same values during the motion. This is permissible since, as indicated earlier, the transformation of \( \omega \) into \( \dot{H}_G \) is independent of the system of coordinate axes selected.

The angular velocity \( \omega \), however, should still be defined with respect to the frame \( GX'Y'Z' \) of fixed orientation. The vector \( \omega \) may then be resolved into components along the rotating \( x, y, z \) axes. Applying the relations (18.7), we obtain the components of the vector \( \dot{H}_G \) along the rotating axes. The vector \( \dot{H}_G \), however, represents the angular momentum about \( G \) of the body in its motion relative to the frame \( GX'Y'Z' \).

Differentiating with respect to \( t \) the components of the angular momentum in (18.7), we define the rate of change of the vector \( H_G \) with respect to the rotating frame \( Gxyz \):

\[ (\dot{H}_G)_{Gxyz} = \dot{H}_i \mathbf{i} + \dot{H}_j \mathbf{j} + \dot{H}_k \mathbf{k} \]  \hspace{1cm} (18.21)

where \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) are the unit vectors along the rotating axes. Recalling from Sec. 15.10 that the rate of change \( \dot{H}_G \) of the vector \( H_G \) with respect to the frame \( GX'Y'Z' \) is found by adding to \( (\dot{H}_G)_{Gxyz} \) the vector product \( \Omega \times H_G \), where \( \Omega \) denotes the angular velocity of the rotating frame, we write

\[ \dot{H}_G = (\dot{H}_G)_{Gxyz} + \Omega \times H_G \]  \hspace{1cm} (18.22)

where \( H_G \) = angular momentum of body with respect to frame \( GX'Y'Z' \) of fixed orientation

\[ (\dot{H}_G)_{Gxyz} \] = rate of change of \( H_G \) with respect to rotating frame \( Gxyz \), to be computed from the relations (18.7) and (18.21)

\( \Omega \) = angular velocity of rotating frame \( Gxyz \)
Substituting for $\mathbf{H}_G$ from (18.22) into (18.2), we have

$$\sum \mathbf{M}_G = (\mathbf{H}_G)_{Gxyz} + \boldsymbol{\Omega} \times \mathbf{H}_G \quad \text{(18.23)}$$

If the rotating frame is attached to the body, as has been assumed in this discussion, its angular velocity $\mathbf{\Omega}$ is identically equal to the angular velocity $\omega$ of the body. There are many applications, however, where it is advantageous to use a frame of reference which is not actually attached to the body but rotates in an independent manner. For example, if the body considered is axisymmetrical, as in Sample Prob. 18.5 or Sec. 18.9, it is possible to select a frame of reference with respect to which the moments and products of inertia of the body remain constant, but which rotates less than the body itself.† As a result, it is possible to obtain simpler expressions for the angular velocity $\omega$ and the angular momentum $\mathbf{H}_G$ of the body than could have been obtained if the frame of reference had actually been attached to the body. It is clear that in such cases the angular velocity $\mathbf{\Omega}$ of the rotating frame and the angular velocity $\omega$ of the body are different.

*18.6 EULER’S EQUATIONS OF MOTION. EXTENSION OF D’ALEMBERT’S PRINCIPLE TO THE MOTION OF A RIGID BODY IN THREE DIMENSIONS

If the $x$, $y$, and $z$ axes are chosen to coincide with the principal axes of inertia of the body, the simplified relations (18.10) can be used to determine the components of the angular momentum $\mathbf{H}_G$. Omitting the primes from the subscripts, we write

$$\mathbf{H}_G = \mathbf{I}_x \omega_x \mathbf{i} + \mathbf{I}_y \omega_y \mathbf{j} + \mathbf{I}_z \omega_z \mathbf{k} \quad \text{(18.24)}$$

where $\mathbf{I}_x$, $\mathbf{I}_y$, and $\mathbf{I}_z$ denote the principal centroidal moments of inertia of the body. Substituting for $\mathbf{H}_G$ from (18.24) into (18.23) and setting $\mathbf{\Omega} = \omega$, we obtain the three scalar equations

$$\sum M_x = \mathbf{I}_x \omega_x - (\mathbf{I}_y - \mathbf{I}_z) \omega_y \omega_z$$
$$\sum M_y = \mathbf{I}_y \omega_y - (\mathbf{I}_z - \mathbf{I}_x) \omega_z \omega_x$$
$$\sum M_z = \mathbf{I}_z \omega_z - (\mathbf{I}_x - \mathbf{I}_y) \omega_x \omega_y \quad \text{(18.25)}$$

These equations, called Euler’s equations of motion after the Swiss mathematician Leonhard Euler (1707–1783), can be used to analyze the motion of a rigid body about its mass center. In the following sections, however, Eq. (18.23) will be used in preference to Eqs. (18.25), since the former is more general and the compact vectorial form in which it is expressed is easier to remember.

Writing Eq. (18.1) in scalar form, we obtain the three additional equations

$$\sum F_x = m \ddot{x} \quad \sum F_y = m \ddot{y} \quad \sum F_z = m \ddot{z} \quad \text{(18.26)}$$

which, together with Euler’s equations, form a system of six differential equations. Given appropriate initial conditions, these differential equations have solutions which express the angular and translational velocities of the body in terms of time. The extension of d’Alembert’s principle (18.2) to the motion of a rigid body in three dimensions is then valid, and the solutions obtained from the extended equations of motion satisfy the given initial conditions.

†More specifically, the frame of reference will have no spin (see Sec. 18.9).
equations have a unique solution. Thus, the motion of a rigid body in three dimensions is completely defined by the resultant and the moment resultant of the external forces acting on it. This result will be recognized as a generalization of a similar result obtained in Sec. 16.4 in the case of the plane motion of a rigid slab. It follows that in three as well as two dimensions, two systems of forces which are equipollent are also equivalent; that is, they have the same effect on a given rigid body.

\[ \sum \mathbf{M}_O = \mathbf{\dot{H}}_O \]  

**Fig. 18.10**

Considering in particular the system of the external forces acting on a rigid body (Fig. 18.10a) and the system of the effective forces associated with the particles forming the rigid body (Fig. 18.10b), we can state that the two systems—which were shown in Sec. 14.2 to be equipollent—are also equivalent. This is the extension of d'Alembert's principle to the three-dimensional motion of a rigid body. Replacing the effective forces in Fig. 18.10b by an equivalent force-couple system, we verify that the system of the external forces acting on a rigid body in three-dimensional motion is equivalent to the system consisting of the vector \( \mathbf{m} \mathbf{a} \) attached at the mass center \( G \) of the body and the couple of moment \( \mathbf{H}_G \) (Fig. 18.11), where \( \mathbf{H}_G \) is obtained from the relations (18.7) and (18.22). Note that the equivalence of the systems of vectors shown in Fig. 18.10 and in Fig. 18.11 has been indicated by red equals signs. Problems involving the three-dimensional motion of a rigid body can be solved by considering the free-body-diagram equation represented in Fig. 18.11 and writing appropriate scalar equations relating the components or moments of the external and effective forces (see Sample Prob. 18.3).

**18.7 Motion of a Rigid Body about a Fixed Point**

When a rigid body is constrained to rotate about a fixed point \( O \), it is desirable to write an equation involving the moments about \( O \) of the external and effective forces, since this equation will not contain the unknown reaction at \( O \). While such an equation can be obtained from Fig. 18.11, it may be more convenient to write it by considering the rate of change of the angular momentum \( \mathbf{H}_O \) of the body about the fixed point \( O \) (Fig. 18.12). Recalling Eq. (14.11), we write

\[ \sum \mathbf{M}_O = \mathbf{\dot{H}}_O \]  

**Fig. 18.11**

**Fig. 18.12**
enables us to relate \( \dot{H}_O \) to the rate of change \( (\dot{H}_O)_{Oxyz} \) of \( H_O \) with respect to the rotating frame \( Oxyz \). Substitution into (18.27) leads to the equation

\[
\Sigma M_O = (\dot{H}_O)_{Oxyz} + \Omega \times H_O
\]

(18.28)

where \( \Sigma M_O \) = sum of moments about \( O \) of forces applied to rigid body

\( H_O \) = angular momentum of body with respect to fixed frame \( OXYZ \)

\( (\dot{H}_O)_{Oxyz} \) = rate of change of \( H_O \) with respect to rotating frame \( Oxyz \), to be computed from relations (18.13)

\( \Omega \) = angular velocity of rotating frame \( Oxyz \)

If the rotating frame is attached to the body, its angular velocity \( \Omega \) is identically equal to the angular velocity \( \omega \) of the body. However, as indicated in the last paragraph of Sec. 18.5, there are many applications where it is advantageous to use a frame of reference which is not actually attached to the body but rotates in an independent manner.

**18.8  ROTATION OF A RIGID BODY ABOUT A FIXED AXIS**

Equation (18.28), which was derived in the preceding section, will be used to analyze the motion of a rigid body constrained to rotate about a fixed axis \( AB \) (Fig. 18.13). First, we note that the angular velocity of the body with respect to the fixed frame \( OXYZ \) is represented by the vector \( \omega \) directed along the axis of rotation. Attaching the moving frame of reference \( Oxyz \) to the body, with the \( z \) axis along \( AB \), we have \( \omega = \omega \hat{k} \). Substituting \( \omega_z = 0, \omega_y = 0, \omega_x = \omega \) into the relations (18.13), we obtain the components along the rotating axes of the angular momentum \( H_O \) of the body about \( O \):

\[
H_x = -I_{xz} \omega \\
H_y = -I_{yz} \omega \\
H_z = I_z \omega
\]

Since the frame \( Oxyz \) is attached to the body, we have \( \Omega = \omega \) and Eq. (18.28) yields

\[
\Sigma M_O = (\dot{H}_O)_{Oxyz} + \omega \times H_O
\]

\[
= (-I_{xz} \omega - I_{yz} \omega) + \omega \times (I_{xz} \omega - I_{yz} \omega) + (I_{zz} \omega - I_{yz} \omega) + (I_{zz} \omega - I_{xz} \omega)
\]

The result obtained can be expressed by the three scalar equations

\[
\Sigma M_x = -I_{xz} \alpha + I_{yz} \omega^2
\]

\[
\Sigma M_y = -I_{yz} \alpha - I_{xz} \omega^2
\]

\[
\Sigma M_z = I_z \alpha
\]

(18.29)

When the forces applied to the body are known, the angular acceleration \( \alpha \) can be obtained from the last of Eqs. (18.29). The angular velocity \( \omega \) is then determined by integration and the values obtained for \( \alpha \) and \( \omega \) substituted into the first two equations (18.29). These equations plus the three equations (18.26) which define the motion of the mass center of the body can then be used to determine the reactions at the bearings \( A \) and \( B \).
It is possible to select axes other than the ones shown in Fig. 18.13 to analyze the rotation of a rigid body about a fixed axis. In many cases, the principal axes of inertia of the body will be found more advantageous. It is therefore wise to revert to Eq. (18.28) and to select the system of axes which best fits the problem under consideration.

If the rotating body is symmetrical with respect to the \( xy \) plane, the products of inertia \( I_{xz} \) and \( I_{yz} \) are equal to zero and Eqs. (18.29) reduce to

\[
\sum M_x = 0 \quad \sum M_y = 0 \quad \sum M_z = I_z \alpha \quad (18.30)
\]

which is in accord with the results obtained in Chap. 16. If, on the other hand, the products of inertia \( I_{xz} \) and \( I_{yz} \) are different from zero, the sum of the moments of the external forces about the \( x \) and \( y \) axes will also be different from zero, even when the body rotates at a constant rate \( \omega \). Indeed, in the latter case, Eqs. (18.29) yield

\[
\sum M_x = I_{yz} \omega^2 \quad \sum M_y = -I_{xz} \omega^2 \quad \sum M_z = 0 \quad (18.31)
\]

This last observation leads us to discuss the balancing of rotating shafts. Consider, for instance, the crankshaft shown in Fig. 18.14a, which is symmetrical about its mass center \( G \). We first observe that when the crankshaft is at rest, it exerts no lateral thrust on its supports, since its center of gravity \( G \) is located directly above \( A \). The shaft is said to be statically balanced. The reaction at \( A \), often referred to as a static reaction, is vertical and its magnitude is equal to the weight \( W \) of the shaft. Let us now assume that the shaft rotates with a constant angular velocity \( \omega \). Attaching our frame of reference to the shaft, with its origin at \( G \), the \( z \) axis along \( AB \), and the \( y \) axis in the plane of symmetry of the shaft (Fig. 18.14b), we note that \( I_{yz} \) is zero and that \( I_{xz} \) is positive. According to Eqs. (18.31), the external forces include a couple of moment \( I_{yz} \omega^2 \). Since this couple is formed by the reaction at \( B \) and the horizontal component of the reaction at \( A \), we have

\[
A_y = \frac{I_{yz} \omega^2}{l} \mathbf{j} \quad B = -\frac{I_{xz} \omega^2}{l} \mathbf{j} \quad (18.32)
\]

Since the bearing reactions are proportional to \( \omega^2 \), the shaft will have a tendency to tear away from its bearings when rotating at high speeds. Moreover, since the bearing reactions \( A_y \) and \( B \), called dynamic reactions, are contained in the \( yz \) plane, they rotate with the shaft and cause the structure supporting it to vibrate. These undesirable effects will be avoided if, by rearranging the distribution of mass around the shaft or by adding corrective masses, we let \( I_{yz} \) become equal to zero. The dynamic reactions \( A_y \) and \( B \) will vanish and the reactions at the bearings will reduce to the static reaction \( A_z \), the direction of which is fixed. The shaft will then be dynamically as well as statically balanced.
SAMPLE PROBLEM 18.3

A slender rod $AB$ of length $L = 8$ ft and weight $W = 40$ lb is pinned at $A$ to a vertical axle $DE$ which rotates with a constant angular velocity $\omega$ of 15 rad/s. The rod is maintained in position by means of a horizontal wire $BC$ attached to the axle and to the end $B$ of the rod. Determine the tension in the wire and the reaction at $A$.

SOLUTION

The effective forces reduce to the vector $m\vec{a}$ attached at $G$ and the couple $H_G$. Since $G$ describes a horizontal circle of radius $\tau = \frac{1}{2}L \cos \beta$ at the constant rate $\omega$, we have

$$\vec{a} = a_x = -\tau \omega^2 \mathbf{i} = -\left(\frac{1}{2}L \cos \beta\right) \omega^2 \mathbf{i} = -\left(450 \text{ ft/s}^2\right) \mathbf{i}$$

$$m\vec{a} = \frac{40}{g} \left(-450\mathbf{i}\right) = -\left(559 \text{ lb}\right) \mathbf{i}$$

Determination of $H_G$. We first compute the angular momentum $H_G$. Using the principal centroidal axes of inertia $x$, $y$, $z$, we write

$$\vec{I}_x = \frac{1}{2}mL^2 \quad \vec{I}_y = 0 \quad \vec{I}_z = \frac{1}{12}mL^2$$

$$\omega_x = -\omega \cos \beta \quad \omega_y = \omega \sin \beta \quad \omega_z = 0$$

$$H_G = \vec{I}_x \omega_x + \vec{I}_y \omega_y + \vec{I}_z \omega_z$$

$$H_G = \frac{1}{12}mL^2 \omega \cos \beta \mathbf{i}$$

The rate of change $\dot{H}_G$ of $H_G$ with respect to axes of fixed orientation is obtained from Eq. (18.22). Observing that the rate of change $(\dot{H}_G)_{Gxyz}$ of $H_G$ with respect to the rotating frame $Gxyz$ is zero, and that the angular velocity $\Omega$ of that frame is equal to the angular velocity $\omega$ of the rod, we have

$$\dot{H}_G = (\dot{H}_G)_{Gxyz} + \omega \times H_G$$

$$\dot{H}_G = 0 + \left(-\omega \cos \beta \mathbf{i} + \omega \sin \beta \mathbf{j}\right) \times \left(-\frac{1}{12}mL^2 \omega \cos \beta \mathbf{i}\right)$$

$$H_G = -\frac{1}{12}mL^2 \omega^2 \sin \beta \cos \beta \mathbf{k} = (645 \text{ lb ft}) \mathbf{k}$$

Equations of Motion. Expressing that the system of the external forces is equivalent to the system of the effective forces, we write

$$\sum M_A = \sum (M_{A})_{eff}$$

$$6.93 \mathbf{j} \times (-7 \mathbf{i} + 2 \mathbf{k} \times (-40 \mathbf{j}) = 3.46 \mathbf{j} \times (-559 \mathbf{i}) + 645 \mathbf{k}$$

$$6.93T - 80 \mathbf{k} = (1934 + 645) \mathbf{k} \quad T = 384 \text{ lb}$$

$$\sum \mathbf{F} = \sum (\mathbf{F})_{eff}$$

$$A_{x} \mathbf{I} + A_{y} \mathbf{J} + A_{z} \mathbf{K} - 384 \mathbf{I} - 40 \mathbf{J} = -559 \mathbf{I}$$

$$A = -\frac{175}{175} \mathbf{I} + (40 \text{ lb}) \mathbf{J}$$

Remark. The value of $T$ could have been obtained from $H_A$ and Eq. (18.28). However, the method used here also yields the reaction at $A$. Moreover, it draws attention to the effect of the asymmetry of the rod on the solution of the problem by clearly showing that both the vector $m\vec{a}$ and the couple $H_G$ must be used to represent the effective forces.
**SAMPLE PROBLEM 18.4**

Two 100-mm rods A and B, each of mass 300 g, are welded to shaft CD which is supported by bearings at C and D. If a couple \( \mathbf{M} \) of magnitude equal to 6 N \( \cdot \) m is applied to the shaft, determine the components of the dynamic reactions at C and D at the instant when the shaft has reached an angular velocity at 1200 rpm. Neglect the moment of inertia of the shaft itself.

---

**SOLUTION**

**Angular Momentum about \( O \).** We attach to the body the frame of reference \( Oxyz \) and note that the axes chosen are not principal axes of inertia for the body. Since the body rotates about the \( x \) axis, we have \( \omega_x = \omega \) and \( \omega_y = \omega_z = 0 \). Substituting into Eqs. (18.13),

\[
\begin{align*}
H_x &= I_x \omega, & H_y &= -I_y \omega, & H_z &= -I_z \omega \\
\mathbf{H}_O &= (I_x \mathbf{i} - I_y \mathbf{j} - I_z \mathbf{k}) \omega
\end{align*}
\]

**Moments of the External Forces about \( O \).** Since the frame of reference rotates with the angular velocity \( \mathbf{\omega} \), Eq. (18.28) yields

\[
\begin{align*}
\Sigma \mathbf{M}_O &= (\mathbf{H}_O)_{Oxyz} + \mathbf{\omega} \times \mathbf{H}_O \\
&= (I_x \mathbf{i} - I_y \mathbf{j} - I_z \mathbf{k}) \alpha + \mathbf{\omega} \times (I_x \mathbf{i} - I_y \mathbf{j} - I_z \mathbf{k}) \omega \\
&= I_x \mathbf{i} \omega_x - (I_y \alpha - I_x \omega_z^2) \mathbf{j} - (I_z \alpha + I_y \omega_x^2) \mathbf{k}
\end{align*}
\]

(1)

**Dynamic Reaction at \( D \).** The external forces consist of the weights of the shaft and rods, the couple \( \mathbf{M} \), the static reactions at \( C \) and \( D \), and the dynamic reactions at \( C \) and \( D \). Since the weights and static reactions are balanced, the external forces reduce to the couple \( \mathbf{M} \) and the dynamic reactions \( \mathbf{C} \) and \( \mathbf{D} \) as shown in the figure. Taking moments about \( O \), we have

\[
\Sigma \mathbf{M}_O = L \mathbf{i} \times (D_j \mathbf{j} + D_k \mathbf{k}) + M \mathbf{i} = M_j \mathbf{i} + D_j \mathbf{j} + D_k \mathbf{k}
\]

(2)

Equating the coefficients of the unit vector \( \mathbf{i} \) in (1) and (2),

\[
M = I_x \alpha, \quad M = 2 \frac{m c \alpha}{L} \alpha, \quad \alpha = 3M/2mc^2
\]

Equating the coefficients of \( \mathbf{k} \) and \( \mathbf{j} \) in (1) and (2):

\[
D_y = -(I_x \alpha + I_y \omega_x^2)/L, \quad D_z = (I_x \alpha - I_y \omega_z^2)/L
\]

(3)

Using the parallel-axis theorem, and noting that the product of inertia of each rod is zero with respect to centroidal axes, we have

\[
I_{xy} = \sum \overline{x} \overline{y} = m \left( \frac{1}{2} L \right) \left( \frac{1}{3} c \right) = \frac{1}{6} mLc \\
I_{xz} = \sum \overline{x} \overline{z} = m \left( \frac{1}{2} L \right) \left( \frac{1}{3} c \right) = \frac{1}{6} mLc
\]

Substituting into (3) the values found for \( I_{xy} \), \( I_{xz} \), and \( \alpha \):

\[
D_y = -\frac{2}{3} (M/c) - \frac{1}{3} mc \omega^2, \quad D_z = \frac{2}{3} (M/c) - \frac{1}{3} mc \omega^2
\]

Substituting \( \omega = 1200 \text{ rpm} = 125.7 \text{ rad/s} \), \( c = 0.100 \text{ m} \), \( M = 6 \text{ N} \cdot \text{m} \), and \( m = 0.300 \text{ kg} \), we have:

\[
D_y = -129.8 \text{ N}, \quad D_z = -36.8 \text{ N}
\]

**Dynamic Reaction at \( C \).** Using a frame of reference attached at \( D \), we obtain equations similar to Eqs. (3), which yield

\[
C_y = -152.2 \text{ N}, \quad C_z = -155.2 \text{ N}
\]
SAMPLE PROBLEM 18.5

A homogeneous disk of radius $r$ and mass $m$ is mounted on an axle $OG$ of length $L$ and negligible mass. The axle is pivoted at the fixed point $O$ and the disk is constrained to roll on a horizontal floor. Knowing that the disk rotates counterclockwise at the constant rate $v_1$ about the axle, determine (a) the force (assumed vertical) exerted by the floor on the disk, (b) the reaction at the pivot $O$.

SOLUTION

The effective forces reduce to the vector $m\vec{a}$ attached at $G$ and the couple $H_G$. Recalling from Sample Prob. 18.2 that the axle rotates about the $y$ axis at the rate $v_2 = r\omega_1/L$, we write

$$m\vec{a} = -mL\omega_1^2\hat{i} = -mL(r\omega_1/L)^2\hat{i} = -(mr^2\omega_1^2/L)\hat{i} \quad (1)$$

Determination of $\dot{H}_G$. We recall from Sample Prob. 18.2 that the angular momentum of the disk about $G$ is

$$H_G = \frac{1}{2}mr^2\omega_1\left(i - \frac{r}{2L}\hat{j}\right)$$

where $H_G$ is resolved into components along the rotating axes $x', y', z'$, with $x'$ along $OG$ and $y'$ vertical. The rate of change $H_G$ of $H_G$ with respect to axes of fixed orientation is obtained from Eq. (18.22). Noting that the rate of change $(\dot{H}_G)_{x'y'z'}$ of $H_G$ with respect to the rotating frame is zero, and that the angular velocity $\Omega$ of that frame is

$$\Omega = -\omega_2\hat{j} = -\frac{r\omega_1}{L}\hat{j}$$

we have

$$\dot{H}_G = (\dot{H}_G)_{x'y'z'} + \Omega \times H_G$$

$$= 0 - \frac{r\omega_1}{L}\hat{j} \times \frac{mr^2\omega_1}{2}
\left(i - \frac{r}{2L}\hat{j}\right)$$

$$= \frac{1}{2}mr^2\omega_1^2\hat{k} \quad (2)$$

Equations of Motion. Expressing that the system of the external forces is equivalent to the system of the effective forces, we write

$$\Sigma\vec{M}_O = \Sigma(M_O)_{eff}$$

$$Li \times (Nj - Wj) = \dot{H}_G$$

$$= \frac{1}{2}mr^2\omega_1^2\hat{k}$$

$$N = W + \frac{1}{2}mr(r/L)^2\omega_1^2\hat{j}$$

$$\Sigma\vec{F} = \Sigma F_{eff}$$

$$\vec{R} + Nj - Wj = m\vec{a}$$

Substituting for $N$ from (3), for $m\vec{a}$ from (1), and solving for $\vec{R}$, we have

$$\vec{R} = -(mr^2\omega_1^2/L)\hat{i} - \frac{1}{2}mr(r/L)^2\omega_1^2\hat{j}$$

$$\vec{R} = -\frac{mr^2\omega_1^2}{L}\left(i + \frac{r}{2L}\hat{j}\right) \quad (3)$$
In this lesson you will be asked to solve problems involving the three-dimensional motion of rigid bodies. The method you will use is basically the same that you used in Chap. 16 in your study of the plane motion of rigid bodies. You will draw a free-body-diagram equation showing that the system of the external forces is equivalent to the system of the effective forces, and you will equate sums of components and sums of moments on both sides of this equation. Now, however, the system of the effective forces will be represented by the vector $\mathbf{ma}$ and a couple vector $\mathbf{H}_G$, the determination of which will be explained in paragraphs 1 and 2 below.

To solve a problem involving the three-dimensional motion of a rigid body, you should take the following steps:

1. Determine the angular momentum $\mathbf{H}_G$ of the body about its mass center $G$ from its angular velocity $\mathbf{\omega}$ with respect to a frame of reference $G\hat{x}Y\hat{Z}$ of fixed orientation. This is an operation you learned to perform in the preceding lesson. However, since the configuration of the body will be changing with time, it will now be necessary for you to use an auxiliary system of axes $G\hat{x}'\hat{y}'\hat{z}'$ (Fig. 18.9) to compute the components of $\mathbf{\omega}$ and the moments and products of inertia of the body. These axes may be rigidly attached to the body, in which case their angular velocity is equal to $\mathbf{\omega}$ [Sample Probs. 18.3 and 18.4], or they may have an angular velocity $\mathbf{\Omega}$ of their own [Sample Prob. 18.5].

Recall the following from the preceding lesson:

   a. If the principal axes of inertia of the body at $G$ are known, use these axes as coordinate axes $x', y',$ and $z'$, since the corresponding products of inertia of the body will be equal to zero. (Note that if the body is axisymmetric, these axes do not need to be rigidly attached to the body.) Resolve $\mathbf{\omega}$ into components $\omega_{x'}, \omega_{y'},$ and $\omega_{z'}$ along these axes and compute the principal moments of inertia $I_{x'}, I_{y'},$ and $I_{z'}$. The corresponding components of the angular momentum $\mathbf{H}_G$ are

   \[
   H_{x'} = I_{x'}\omega_{x'} \quad H_{y'} = I_{y'}\omega_{y'} \quad H_{z'} = I_{z'}\omega_{z'}
   \]  

   (18.10)

   b. If the principal axes of inertia of the body at $G$ are not known, you must use Eqs. (18.7) to determine the components of the angular momentum $\mathbf{H}_G$. These equations require your prior computation of the products of inertia of the body, as well as of its moments of inertia, with respect to the selected axes.

(continued)
2. Compute the rate of change \( \dot{H}_G \) of the angular momentum \( H_G \) with respect to the frame \( G'X'Y'Z' \). Note that this frame has a fixed orientation, while the frame \( Gx'y'z' \) you used when you calculated the components of the vector \( \omega \) was a rotating frame. We refer you to our discussion in Sec. 15.10 of the rate of change of a vector with respect to a rotating frame. Recalling Eq. (15.31), you will express the rate of change \( \dot{H}_G \) as follows:

\[
\dot{H}_G = (\dot{H}_G)_{G'X'Y'} + \Omega \times H_G
\]  
(18.22)

The first term in the right-hand member of Eq. (18.22) represents the rate of change of \( H_G \) with respect to the rotating frame \( G'x'y'z' \). This term will drop out if \( \omega \)—and, thus, \( H_G \)—remain constant in both magnitude and direction when viewed from that frame. On the other hand, if any of the time derivatives \( \dot{\omega}_x \), \( \dot{\omega}_y \), and \( \dot{\omega}_z \) is different from zero, \( (\dot{H}_G)_{G'X'Y'} \) will also be different from zero, and its components should be determined by differentiating Eqs. (18.10) with respect to \( t \). Finally, we remind you that if the rotating frame is rigidly attached to the body, its angular velocity will be the same as that of the body, and \( \Omega \) can be replaced by \( \omega \).

3. Draw the free-body-diagram equation for the rigid body, showing that the system of the external forces exerted on the body is equivalent to the vector \( m\dot{a} \) applied at \( G \) and the couple vector \( H_G \) (Fig. 18.11). By equating components in any direction and moments about any point, you can write as many as six independent scalar equations of motion [Sample Probs. 18.3 and 18.5].

4. When solving problems involving the motion of a rigid body about a fixed point \( O \), you may find it convenient to use the following equation, derived in Sec. 18.7, which eliminates the components of the reaction at the support \( O \).

\[
\sum \mathbf{M}_O = (\dot{H}_O)_{Oxyz} + \Omega \times H_O
\]  
(18.28)

where the first term in the right-hand member represents the rate of change of \( H_O \) with respect to the rotating frame \( Oxyz \), and where \( \Omega \) is the angular velocity of that frame.

5. When determining the reactions at the bearings of a rotating shaft, use Eq. (18.28) and take the following steps:

a. Place the fixed point \( O \) at one of the two bearings supporting the shaft and attach the rotating frame \( Oxyz \) to the shaft, with one of the axes directed along it. Assuming, for instance, that the \( x \) axis has been aligned with the shaft, you will have \( \Omega = \omega = \omega \hat{i} \) [Sample Prob. 18.4].
b. Since the selected axes, usually, will not be the principal axes of inertia at \( O \), you must compute the products of inertia of the shaft, as well as its moments of inertia, with respect to these axes, and use Eqs. (18.13) to determine \( \mathbf{H}_O \). Assuming again that the \( x \) axis has been aligned with the shaft, Eqs. (18.13) reduce to

\[
H_x = I_x \omega \\
H_y = -I_y \omega \\
H_z = -I_z \omega
\]

which shows that \( \mathbf{H}_O \) will not be directed along the shaft.

c. To obtain \( \dot{\mathbf{H}}_O \), substitute the expressions obtained into Eq. (18.28), and let \( \mathbf{\Omega} = \omega = \omega \mathbf{i} \). If the angular velocity of the shaft is constant, the first term in the right-hand member of the equation will drop out. However, if the shaft has an angular acceleration \( \mathbf{\alpha} = \dot{\omega} \mathbf{i} \), the first term will not be zero and must be determined by differentiating with respect to \( t \) the expressions in (18.13'). The result will be equations similar to Eqs. (18.13'), with \( \omega \) replaced by \( \mathbf{\alpha} \).

d. Since point \( O \) coincides with one of the bearings, the three scalar equations corresponding to Eq. (18.28) can be solved for the components of the dynamic reaction at the other bearing. If the mass center \( G \) of the shaft is located on the line joining the two bearings, the effective force \( m\mathbf{\alpha} \) will be zero. Drawing the free-body-diagram equation of the shaft, you will then observe that the components of the dynamic reaction at the first bearing must be equal and opposite to those you have just determined. If \( G \) is not located on the line joining the two bearings, you can determine the reaction at the first bearing by placing the fixed point \( O \) at the second bearing and repeating the earlier procedure [Sample Prob. 18.4]; or you can obtain additional equations of motion from the free-body-diagram equation of the shaft, making sure to first determine and include the effective force \( m\mathbf{\alpha} \) applied at \( G \).

e. Most problems call for the determination of the “dynamic reactions” at the bearings, that is, for the additional forces exerted by the bearings on the shaft when the shaft is rotating. When determining dynamic reactions, ignore the effect of static loads, such as the weight of the shaft.
18.55 Determine the rate of change $\mathbf{\dot{H}}_D$ of the angular momentum $\mathbf{H}_D$ of the assembly of Prob. 18.1.

18.56 Determine the rate of change $\mathbf{\dot{H}}_C$ of the angular momentum $\mathbf{H}_C$ of the disk of Prob. 18.2.

18.57 Determine the rate of change $\mathbf{\dot{H}}_A$ of the angular momentum $\mathbf{H}_A$ of the plate of Prob. 18.3, knowing that its angular velocity $\mathbf{\omega}$ remains constant.

18.58 Determine the rate of change $\mathbf{\dot{H}}_G$ of the angular momentum $\mathbf{H}_G$ of the disk of Prob. 18.4.

18.59 Determine the rate of change $\mathbf{\dot{H}}_G$ of the angular momentum $\mathbf{H}_G$ of the disk of Prob. 18.5.

18.60 Determine the rate of change $\mathbf{\dot{H}}_A$ of the angular momentum $\mathbf{H}_A$ of the disk of Prob. 18.6.

18.61 Determine the rate of change $\mathbf{\dot{H}}_D$ of the angular momentum $\mathbf{H}_D$ of the assembly of Prob. 18.1, assuming that at the instant considered the assembly has an angular velocity $\mathbf{\omega} = (12 \text{ rad/s})\mathbf{i}$ and an angular acceleration $\mathbf{\alpha} = (96 \text{ rad/s}^2)\mathbf{i}$.

18.62 Determine the rate of change $\mathbf{\dot{H}}_D$ of the angular momentum $\mathbf{H}_D$ of the assembly of Prob. 18.1, assuming that at the instant considered the assembly has an angular velocity $\mathbf{\omega} = (12 \text{ rad/s})\mathbf{i}$ and an angular acceleration $\mathbf{\alpha} = -(96 \text{ rad/s}^2)\mathbf{i}$.

18.63 Determine the rate of change $\mathbf{\dot{H}}_A$ of the angular momentum $\mathbf{H}_A$ of the plate of Prob. 18.3, assuming that it has an angular velocity $\mathbf{\omega} = \omega_j$ and an angular acceleration $\mathbf{\alpha} = \alpha_j$.

18.64 Determine the rate of change $\mathbf{\dot{H}}_G$ of the angular momentum $\mathbf{H}_G$ of the disk of Prob. 18.4, assuming that at the instant considered the assembly has an angular velocity $\mathbf{\omega} = \omega_j$ and an angular acceleration $\mathbf{\alpha} = \alpha_j$.

18.65 A thin homogeneous triangular plate of mass 2.5 kg is welded to a light vertical axle supported by bearings at A and B. Knowing that the plate rotates at the constant rate $\omega = 8 \text{ rad/s}$, determine the dynamic reactions at A and B.

18.66 A slender, uniform rod AB of mass $m$ and a vertical shaft CD, each of length $2b$, are welded together at their midpoints G. Knowing that the shaft rotates at the constant rate $\omega$, determine the dynamic reactions at C and D.
18.67 The 16-lb shaft shown has a uniform cross section. Knowing that the shaft rotates at the constant rate \( \omega = 12 \text{ rad/s} \), determine the dynamic reactions at A and B.

18.68 The assembly shown consists of pieces of sheet aluminum of uniform thickness and of total weight 2.7 lb welded to a light axle supported by bearings at A and B. Knowing that the assembly rotates at the constant rate \( \omega = 240 \text{ rpm} \), determine the dynamic reactions at A and B.

![Fig. P18.67](image)

18.69 When the 18-kg wheel shown is attached to a balancing machine and made to spin at a rate of 12.5 rev/s, it is found that the forces exerted by the wheel on the machine are equivalent to a force-couple system consisting of a force \( F = (160 \text{ N}) \hat{j} \) applied at C and a couple \( \mathbf{M}_C = (14.7 \text{ N} \cdot \text{m}) \hat{k} \), where the unit vectors form a triad which rotates with the wheel. (a) Determine the distance from the axis of rotation to the mass center of the wheel and the products of inertia \( I_{xy} \) and \( I_{zx} \). (b) If only two corrective masses are to be used to balance the wheel statically and dynamically, what should these masses be and at which of the points A, B, D, or E should they be placed?

18.70 After attaching the 18-kg wheel shown to a balancing machine and making it spin at the rate of 15 rev/s, a mechanic has found that to balance the wheel both statically and dynamically, he should use two corrective masses, a 170-g mass placed at B and a 56-g mass placed at D. Using a right-handed frame of reference rotating with the \( z \)-axis perpendicular to the plane of the figure, determine before the corrective masses have been attached (a) the distance from the axis of rotation to the mass center of the wheel and the products of inertia \( I_{xy} \) and \( I_{zx} \), (b) the force-couple system at C equivalent to the forces exerted by the wheel on the machine.

18.71 Knowing that the plate of Prob. 18.65 is initially at rest (\( \omega = 0 \)) when a couple of moment \( \mathbf{M}_0 = (0.75 \text{ N} \cdot \text{m}) \hat{j} \) is applied to it, determine (a) the resulting angular acceleration of the plate, (b) the dynamics reactions A and B immediately after the couple has been applied.

18.72 Knowing that the assembly of Prob. 18.66 is initially at rest (\( \omega = 0 \)) when a couple of moment \( \mathbf{M}_0 = M_0 \hat{j} \) is applied to shaft CD, determine (a) the resulting angular acceleration of the assembly, (b) the dynamic reactions at C and D immediately after the couple is applied.
18.73 The sheet-metal component shown is of uniform thickness and has a mass of 600 g. It is attached to a light axle supported by bearings at A and B located 150 mm apart. The component is at rest when it is subjected to a couple $M_0$ as shown. If the resulting angular acceleration is $\alpha = (12 \text{ rad/s}^2)\hat{k}$, determine (a) the couple $M_0$, (b) the dynamic reactions at A and B immediately after the couple has been applied.

18.74 For the sheet-metal component of Prob. 18.73, determine (a) the angular velocity of the component 0.6 s after the couple $M_0$ has been applied to it, (b) the magnitude of the dynamic reactions at A and B at that time.

18.75 The shaft of Prob. 18.67 is initially at rest ($\omega = 0$) when a couple $M_0$ is applied to it. Knowing that the resulting angular acceleration of the shaft is $\alpha = (20 \text{ rad/s}^2)\hat{i}$, determine (a) the couple $M_0$, (b) the dynamic reactions at A and B immediately after the couple is applied.

18.76 The assembly of Prob. 18.68 is initially at rest ($\omega = 0$) when a couple $M_0$ is applied to axle AB. Knowing that the resulting angular acceleration of the assembly is $\alpha = (150 \text{ rad/s}^2)\hat{i}$, determine (a) the couple $M_0$, (b) the dynamic reactions at A and B immediately after the couple is applied.

18.77 The assembly shown weighs 12 lb and consists of 4 thin 16-in.-diameter semicircular aluminum plates welded to a light 40-in.-long shaft AB. The assembly is at rest ($\omega = 0$) at time $t = 0$ when a couple $M_0$ is applied to it as shown, causing the assembly to complete one full revolution in 2 s. Determine (a) the couple $M_0$, (b) the dynamic reactions at A and B at $t = 0$.

18.78 For the assembly of Prob. 18.77, determine the dynamic reactions at A and B at $t = 2$ s.
18.79 The flywheel of an automobile engine, which is rigidly attached to the crankshaft, is equivalent to a 400-mm-diameter, 15-mm-thick steel plate. Determine the magnitude of the couple exerted by the flywheel on the horizontal crankshaft as the automobile travels around an unbanked curve of 200-m radius at a speed of 90 km/h, with the flywheel rotating at 2700 rpm. Assume the automobile to have (a) a rear-wheel drive with the engine mounted longitudinally, (b) a front-wheel drive with the engine mounted transversely. (Density of steel = 7860 kg/m$^3$.)

18.80 A four-bladed airplane propeller has a mass of 160 kg and a radius of gyration of 800 mm. Knowing that the propeller rotates at 1600 rpm as the airplane is traveling in a circular path of 600-m radius at 540 km/h, determine the magnitude of the couple exerted by the propeller on its shaft due to the rotation of the airplane.

18.81 The blade of a portable saw and the rotor of its motor have a total weight of 2.5 lb and a combined radius of gyration of 1.5 in. Knowing that the blade rotates as shown at the rate $\omega_1 = 1500$ rpm, determine the magnitude and direction of the couple $\mathbf{M}$ that a worker must exert on the handle of the saw to rotate it with a constant angular velocity $\omega_2 = -\langle 2.4 \text{ rad/s} \rangle \hat{j}$.

18.82 The blade of an oscillating fan and the rotor of its motor have a total weight of 8 oz and a combined radius of gyration of 3 in. They are supported by bearings at A and B, 5 in. apart, and rotate at the rate $\omega_1 = 1800$ rpm. Determine the dynamic reactions at A and B when the motor casing has an angular velocity $\omega_2 = \langle 0.6 \text{ rad/s} \rangle \hat{j}$.

18.83 Each wheel of an automobile has a mass of 22 kg, a diameter of 575 mm, and a radius of gyration of 225 mm. The automobile travels around an unbanked curve of radius 150 m at a speed of 95 km/h. Knowing that the transverse distance between the wheels is 1.5 m, determine the additional normal force exerted by the ground on each outside wheel due to the motion of the car.

18.84 The essential structure of a certain type of aircraft turn indicator is shown. Each spring has a constant of 500 N/m, and the 200-g uniform disk of 40-mm radius spins at the rate of 10 000 rpm. The springs are stretched and exert equal vertical forces on yoke AB when the airplane is traveling in a straight path. Determine the angle through which the yoke will rotate when the pilot executes a horizontal turn of 750-m radius to the right at a speed of 800 km/h. Indicate whether point A will move up or down.
18.85 A uniform semicircular plate of radius 120 mm is hinged at A and B to a clevis which rotates with a constant angular velocity \( \omega \) about a vertical axis. Determine (a) the angle \( \beta \) that the plate forms with the horizontal x axis when \( \omega = 15 \text{ rad/s} \), (b) the largest value of \( \omega \) for which the plate remains vertical (\( \beta = 90^\circ \)).

18.86 A uniform semicircular plate of radius 120 mm is hinged at A and B to a clevis which rotates with a constant angular velocity \( \omega \) about a vertical axis. Determine the value of \( \omega \) for which the plate forms an angle \( \beta = 50^\circ \) with the horizontal x axis.

18.87 A slender rod is bent to form a square frame of side 6 in. The frame is attached by a collar at A to a vertical shaft which rotates with a constant angular velocity \( \omega \). Determine (a) the angle \( \beta \) that line AB forms with the horizontal x axis when \( \omega = 9.8 \text{ rad/s} \), (b) the largest value of \( \omega \) for which \( \beta = 90^\circ \).

18.88 A slender rod is bent to form a square frame of side 6 in. The frame is attached by a collar at A to a vertical shaft which rotates with a constant angular velocity \( \omega \). Determine the value of \( \omega \) for which line AB forms an angle \( \beta = 48^\circ \) with the horizontal x axis.

18.89 The 950-g gear A is constrained to roll on the fixed gear B, but is free to rotate about axle AD. Axle AD, of length 400 mm and negligible mass, is connected by a clevis to the vertical shaft DE which rotates as shown with a constant angular velocity \( \omega_1 \). Assuming that gear A can be approximated by a thin disk of radius 80 mm, determine the largest allowable value of \( \omega_1 \) if gear A is not to lose contact with gear B.

18.90 Determine the force \( \mathbf{F} \) exerted by gear B on gear A of Prob. 18.89 when shaft DE rotates with the constant angular velocity \( \omega_1 = 4 \text{ rad/s} \). (Hint: The force \( \mathbf{F} \) must be perpendicular to the line drawn from D to C.)
18.91 and 18.92 The slender rod $AB$ is attached by a clevis to arm $BCD$ which rotates with a constant angular velocity $\omega$ about the centerline of its vertical portion $CD$. Determine the magnitude of the angular velocity $\omega$.

![Diagram of Problem 18.91 and 18.92]

18.93 Two disks, each of mass 5 kg and radius 100 mm, spin as shown at the rate $\omega_1 = 1500$ rpm about a rod $AB$ of negligible mass which rotates about a vertical axis at the rate $\omega_2 = 45$ rpm. (a) Determine the dynamic reactions at $C$ and $D$. (b) Solve part a assuming that the direction of spin of disk $B$ is reversed.

![Diagram of Problem 18.93 and 18.94]

18.94 Two disks, each of mass 5 kg and radius 100 mm, spin as shown at the rate $\omega_1 = 1500$ rpm about a rod $AB$ of negligible mass which rotates about a vertical axis at a rate $\omega_2$. Determine the maximum allowable value of $\omega_2$ if the dynamic reactions at $C$ and $D$ are not to exceed 250 N each.
18.95 The 10-oz disk shown spins at the rate $\omega_1 = 750$ rpm, while axle $AB$ rotates as shown with an angular velocity $\omega_2$ of 6 rad/s. Determine the dynamic reactions at $A$ and $B$.

18.96 The 10-oz disk shown spins at the rate $\omega_1 = 750$ rpm, while axle $AB$ rotates as shown with an angular velocity $\omega_2$. Determine the maximum allowable magnitude of $\omega_2$ if the dynamic reactions at $A$ and $B$ are not to exceed 0.25 lb each.

18.97 A thin disk of weight $W = 10$ lb rotates with an angular velocity $\omega_2$ with respect to arm $ABC$, which itself rotates with an angular velocity $\omega_1$ about the $y$ axis. Knowing that $\omega_1 = 5$ rad/s and $\omega_2 = 15$ rad/s and that both are constant, determine the force-couple system representing the dynamic reaction at the support at $A$.

18.98 A homogeneous disk of weight $W = 6$ lb rotates at the constant rate $\omega_1 = 16$ rad/s with respect to arm $ABC$, which is welded to a shaft $DCE$ rotating at the constant rate $\omega_2 = 8$ rad/s. Determine the dynamic reactions at $D$ and $E$.

*18.99 A 48-kg advertising panel of length $2a = 2.4$ m and width $2b = 1.6$ m is kept rotating at a constant rate $\omega_1$ about its horizontal axis by a small electric motor attached at $A$ to frame $ACB$. This frame itself is kept rotating at a constant rate $\omega_2$ about a vertical axis by a second motor attached at $C$ to the column $CD$. Knowing that the panel and the frame complete a full revolution in $6$ s and $12$ s, respectively, express, as a function of the angle $\theta$, the dynamic reaction exerted on column $CD$ by its support at $D$. 
18.100 For the system of Prob. 18.99, show that (a) the dynamic reaction at \( D \) is independent of the length \( 2a \) of the panel, (b) the ratio \( M_1/M_2 \) of the magnitudes of the couples exerted by the motors at \( A \) and \( C \), respectively, is independent of the dimensions and mass of the panel and is equal to \( \omega_2/2\omega_1 \) at any given instant.

18.101 A 3-kg homogeneous disk of radius 60 mm spins as shown at the constant rate \( \omega_1 = 60 \) rad/s. The disk is supported by the fork-ended rod \( AB \), which is welded to the vertical shaft \( CBD \). The system is at rest when a couple \( M_0 = (0.40 \text{ N m}) \) is applied to the shaft for 2 s and then removed. Determine the dynamic reactions at \( C \) and \( D \) after the couple has been removed.

18.102 A 3-kg homogeneous disk of radius 60 mm spins as shown at the constant rate \( \omega_1 = 5 \) rad/s. The disk is supported by the fork-ended rod \( AB \), which is welded to the vertical shaft \( CBD \). The system is at rest when a couple \( M_0 \) is applied as shown to the shaft for 3 s and then removed. Knowing that the maximum angular velocity reached by the shaft is 18 rad/s, determine (a) the couple \( M_0 \), (b) the dynamic reactions at \( C \) and \( D \) after the couple has been removed.

18.103 For the disk of Prob. 18.97, determine (a) the couple \( M_1 \) which should be applied to arm \( ABC \) to give it an angular acceleration \( \alpha_1 = -(7.5 \text{ rad/s}^2)j \) when \( \omega_1 = 5 \text{ rad/s} \), knowing that the disk rotates at the constant rate \( \omega_2 = 15 \text{ rad/s} \), (b) the force-couple system representing the dynamic reaction at \( A \) at that instant. Assume that \( ABC \) has a negligible mass.

18.104 It is assumed that at the instant shown shaft \( DCE \) of Prob. 18.98 has an angular velocity \( \omega_2 = (8 \text{ rad/s})i \) and an angular acceleration \( \alpha_2 = (6 \text{ rad/s}^2)i \). Recalling that the disk rotates with a constant angular velocity \( \omega_1 = (16 \text{ rad/s})j \), determine (a) the couple which must be applied to shaft \( DCE \) to produce the given angular acceleration, (b) the corresponding dynamic reactions at \( D \) and \( E \).

18.105 A 2.5-kg homogeneous disk of radius 80 mm rotates with an angular velocity \( \omega_1 \) with respect to arm \( ABC \), which is welded to a shaft \( DCE \) rotating as shown at the constant rate \( \omega_2 = 12 \text{ rad/s} \). Friction in the bearing at \( A \) causes \( \omega_1 \) to decrease at the rate of 15 rad/s\(^2\). Determine the dynamic reactions at \( D \) and \( E \) at a time when \( \omega_1 \) has decreased to 50 rad/s.

18.106 A slender homogeneous rod \( AB \) of mass \( m \) and length \( L \) is made to rotate at a constant rate \( \omega_2 \) about the horizontal \( z \) axis, while frame \( CD \) is made to rotate at the constant rate \( \omega_1 \) about the \( y \) axis. Express as a function of the angle \( \theta \) (a) the couple \( M_1 \) required to maintain the rotation of the frame, (b) the couple \( M_2 \) required to maintain the rotation of the rod, (c) the dynamic reactions at the supports \( C \) and \( D \).
**18.9 MOTION OF A GYROSCOPE. EULERIAN ANGLES**

A gyroscope consists essentially of a rotor which can spin freely about its geometric axis. When mounted in a Cardan’s suspension (Fig. 18.15), a gyroscope can assume any orientation, but its mass center must remain fixed in space. In order to define the position of a gyroscope at a given instant, let us select a fixed frame of reference \( OXYZ \), with the origin \( O \) located at the mass center of the gyroscope and the \( Z \) axis directed along the line defined by the bearings \( A \) and \( A' \) of the outer gimbal. We will consider a reference position of the gyroscope in which the two gimbals and a given diameter \( DD' \) of the rotor are located in the fixed \( YZ \) plane (Fig. 18.15a). The gyroscope can be brought from this reference position into any arbitrary position (Fig. 18.15b) by means of the following steps: (1) a rotation of the outer gimbal through an angle \( \phi \) about the axis \( AA' \), (2) a rotation of the inner gimbal through \( \theta \) about \( BB' \), and (3) a rotation of the rotor through \( \psi \) about \( CC' \). The angles \( \phi, \theta, \) and \( \psi \) are called the Eulerian angles; they completely characterize the position of the gyroscope at any given instant. Their derivatives \( \dot{\phi}, \dot{\theta}, \) and \( \dot{\psi} \) define, respectively, the rate of precession, the rate of nutation, and the rate of spin of the gyroscope at the instant considered.

In order to compute the components of the angular velocity and of the angular momentum of the gyroscope, we will use a rotating system of axes \( Oxyz \) attached to the inner gimbal, with the \( y \) axis along \( BB' \) and the \( z \) axis along \( CC' \) (Fig. 18.16). These axes are principal axes of inertia for the gyroscope. While they follow it in its precession and nutation, however, they do not spin; for that reason, they are more convenient to use than axes actually attached to the gyroscope. The angular velocity \( \omega \) of the gyroscope with respect to the fixed frame of reference \( OXYZ \) will now be expressed as the sum of three partial angular velocities corresponding, respectively, to the precession, the nutation, and the spin of the gyroscope. Denoting by \( \mathbf{i}, \mathbf{j}, \) and \( \mathbf{k} \) the unit vectors along the rotating axes, and by \( \mathbf{K} \) the unit vector along the fixed \( Z \) axis, we have

\[
\omega = \dot{\phi} \mathbf{K} + \dot{\theta} \mathbf{j} + \dot{\psi} \mathbf{k} \quad (18.33)
\]

Since the vector components obtained for \( \omega \) in (18.33) are not orthogonal (Fig. 18.16), the unit vector \( \mathbf{K} \) will be resolved into components along the \( x \) and \( z \) axes; we write

\[
\mathbf{K} = -\sin \theta \mathbf{i} + \cos \theta \mathbf{k} \quad (18.34)
\]

and, substituting for \( \mathbf{K} \) into (18.33),

\[
\omega = -\dot{\phi} \sin \theta \mathbf{i} + \dot{\theta} \mathbf{j} + (\dot{\psi} + \dot{\phi} \cos \theta) \mathbf{k} \quad (18.35)
\]

Since the coordinate axes are principal axes of inertia, the components of the angular momentum \( \mathbf{H}_O \) can be obtained by multiplying
the components of $\omega$ by the moments of inertia of the rotor about the $x$, $y$, and $z$ axes, respectively. Denoting by $I$ the moment of inertia of the rotor about its spin axis, by $I'$ its moment of inertia about a transverse axis through $O$, and neglecting the mass of the gimbals, we write

$$H_O = -I'\dot{\phi} \sin \theta \mathbf{i} + I'\dot{\theta} \mathbf{j} + I(\dot{\psi} + \dot{\phi} \cos \theta) \mathbf{k} \tag{18.36}$$

Recalling that the rotating axes are attached to the inner gimbal and thus do not spin, we express their angular velocity as the sum

$$\Omega = \dot{\phi} \mathbf{K} + \dot{\theta} \mathbf{j} \tag{18.37}$$

or, substituting for $\mathbf{K}$ from (18.34),

$$\Omega = -\dot{\phi} \sin \theta \mathbf{i} + \dot{\theta} \mathbf{j} + \dot{\phi} \cos \theta \mathbf{k} \tag{18.38}$$

Substituting for $H_O$ and $\Omega$ from (18.36) and (18.38) into the equation

$$\sum M_O = (\dot{H}_O)_{Oxyz} + \Omega \times H_O \tag{18.28}$$

we obtain the three differential equations

$$\begin{align*}
\sum M_x &= -I'(-\dot{\phi} \sin \theta + 2\dot{\phi} \dot{\phi} \cos \theta) + I\dot{\theta} (\dot{\psi} + \dot{\phi} \cos \theta) \\
\sum M_y &= I'(-\dot{\theta} - \dot{\phi}^2 \sin \theta \cos \theta) + I\dot{\phi} \sin \theta (\dot{\psi} + \dot{\phi} \cos \theta) \\
\sum M_z &= I \frac{d}{dt} (\dot{\psi} + \dot{\phi} \cos \theta)
\end{align*} \tag{18.39}$$

The equations (18.39) define the motion of a gyroscope subjected to a given system of forces when the mass of its gimbals is neglected. They can also be used to define the motion of an axisymmetrical body (or body of revolution) attached at a point on its axis of symmetry, and to define the motion of an axisymmetrical body about its mass center. While the gimbals of the gyroscope helped us visualize the Eulerian angles, it is clear that these angles can be used to define the position of any rigid body with respect to axes centered at a point of the body, regardless of the way in which the body is actually supported.

Since the equations (18.39) are nonlinear, it will not be possible, in general, to express the Eulerian angles $\phi$, $\theta$, and $\psi$ as analytical functions of the time $t$, and numerical methods of solution may have to be used. However, as you will see in the following sections, there are several particular cases of interest which can be analyzed easily.
\*18.10 STEADY PRECESSION OF A GYROSCOPE

Let us now investigate the particular case of gyroscopic motion in which the angle $\theta$, the rate of precession $\dot{\theta}$, and the rate of spin $\dot{\psi}$ remain constant. We propose to determine the forces which must be applied to the gyroscope to maintain this motion, known as the \textit{steady precession} of a gyroscope.

Instead of applying the general equations (18.39), we will determine the sum of the moments of the required forces by computing the rate of change of the angular momentum of the gyroscope in the particular case considered. We first note that the angular velocity $\omega$ of the gyroscope, its angular momentum $H_O$, and the angular velocity $\Omega$ of the rotating frame of reference (Fig. 18.17) reduce, respectively, to

$$\omega = -\dot{\phi} \sin \theta \mathbf{i} + \omega_z \mathbf{k}$$  \hspace{1cm} (18.40)
$$H_O = -\dot{I}' \dot{\phi} \sin \theta \mathbf{i} + I \omega \mathbf{k}$$  \hspace{1cm} (18.41)
$$\Omega = -\dot{\phi} \sin \theta \mathbf{i} + \dot{\phi} \cos \theta \mathbf{k}$$  \hspace{1cm} (18.42)

where $\omega_z = \dot{\psi} + \dot{\phi} \cos \theta$ = rectangular component along spin axis of total angular velocity of gyroscope.

Since $\theta$, $\phi$, and $\psi$ are constant, the vector $H_O$ is constant in magnitude and direction with respect to the rotating frame of reference and its rate of change $\frac{dH_O}{d\theta}$ with respect to that frame is zero. Thus Eq. (18.28) reduces to

$$\sum M_O = \Omega \times H_O$$  \hspace{1cm} (18.43)

which yields, after substitutions from (18.41) and (18.42),

$$\sum M_O = (I \omega_z - I' \dot{\phi} \cos \theta) \dot{\phi} \sin \theta \mathbf{j}$$  \hspace{1cm} (18.44)

Since the mass center of the gyroscope is fixed in space, we have, by (18.1), $\sum F = 0$; thus, the forces which must be applied to the gyroscope to maintain its steady precession reduce to a couple of moment equal to the right-hand member of Eq. (18.44). We note that \textit{this couple should be applied about an axis perpendicular to the precession axis and to the spin axis of the gyroscope} (Fig. 18.18).

In the particular case when the precession axis and the spin axis are at a right angle to each other, we have $\theta = 90^\circ$ and Eq. (18.44) reduces to

$$\sum M_O = I \dot{\phi} \mathbf{j}$$  \hspace{1cm} (18.45)

Thus, if we apply to the gyroscope a couple $M_O$ about an axis perpendicular to its axis of spin, the gyroscope will precess about an axis perpendicular to both the spin axis and the couple axis, in a sense such that the vectors representing the spin, the couple, and the precession, respectively, form a right-handed triad (Fig. 18.19).

Because of the relatively large couples required to change the orientation of their axles, gyroscopes are used as stabilizers in torpedoes...
and ships. Spinning bullets and shells remain tangent to their trajectory because of gyroscopic action. And a bicycle is easier to keep balanced at high speeds because of the stabilizing effect of its spinning wheels. However, gyroscopic action is not always welcome and must be taken into account in the design of bearings supporting rotating shafts subjected to forced precession. The reactions exerted by its propellers on an airplane which changes its direction of flight must also be taken into consideration and compensated for whenever possible.

*18.11 MOTION OF AN AXISYMMETRICAL BODY UNDER NO FORCE

In this section you will analyze the motion about its mass center of an axisymmetrical body under no force except its own weight. Examples of such a motion are furnished by projectiles, if air resistance is neglected, and by artificial satellites and space vehicles after burnout of their launching rockets.

Since the sum of the moments of the external forces about the mass center \( G \) of the body is zero, Eq. (18.2) yields \( H_G = 0 \). It follows that the angular momentum \( H_G \) of the body about \( G \) is constant. Thus, the direction of \( H_G \) is fixed in space and can be used to define the \( Z \) axis, or axis of precession (Fig. 18.20). Selecting a rotating system of axes \( Gxyz \) with the \( z \) axis along the axis of symmetry of the body, the \( x \) axis in the plane defined by the \( Z \) and \( z \) axes, and the \( y \) axis pointing away from you, we have

\[
H_x = -H_G \sin \theta \\
H_y = 0 \\
H_z = H_G \cos \theta
\]  
(18.46)

where \( \theta \) represents the angle formed by the \( Z \) and \( z \) axes, and \( H_G \) denotes the constant magnitude of the angular momentum of the body about \( G \). Since the \( x \), \( y \), and \( z \) axes are principal axes of inertia for the body considered, we can write

\[
H_x = I'\omega_x \\
H_y = I'\omega_y \\
H_z = I\omega_z
\]  
(18.47)

where \( I \) denotes the moment of inertia of the body about its axis of symmetry and \( I' \) denotes its moment of inertia about a transverse axis through \( G \). It follows from Eqs. (18.46) and (18.47) that

\[
\omega_x = -\frac{H_G \sin \theta}{I'} \\
\omega_y = 0 \\
\omega_z = \frac{H_G \cos \theta}{I}
\]  
(18.48)

The second of the relations obtained shows that the angular velocity \( \omega \) has no component along the \( y \) axis, i.e., along an axis perpendicular to the \( Zz \) plane. Thus, the angle \( \theta \) formed by the \( Z \) and \( z \) axes remains constant and the body is in steady precession about the \( Z \) axis.

Dividing the first and third of the relations (18.48) member by member, and observing from Fig. 18.21 that \( -\omega_x/\omega_z = \tan \gamma \), we obtain the following relation between the angles \( \gamma \) and \( \theta \) that the

\[
\phi = \gamma = \tan \theta
\]  
(18.49)
Kinetics of Rigid Bodies in Three Dimensions

vectors $\mathbf{\omega}$ and $\mathbf{H}_C$, respectively, form with the axis of symmetry of the body:

$$\tan \gamma = \frac{I}{I'} \tan \theta$$  \hspace{1cm} (18.49)

There are two particular cases of motion of an axisymmetrical body under no force which involve no precession: (1) If the body is set to spin about its axis of symmetry, we have $\omega_z = 0$ and, by (18.47), $H_z = 0$; the vectors $\mathbf{\omega}$ and $\mathbf{H}_C$ have the same orientation and the body keeps spinning about its axis of symmetry (Fig. 18.22a). (2) If the body is set to spin about a transverse axis, we have $\omega_z = 0$ and, by (18.47), $H_z = 0$; again $\mathbf{\omega}$ and $\mathbf{H}_C$ have the same orientation and the body keeps spinning about the given transverse axis (Fig. 18.22b).

Considering now the general case represented in Fig. 18.21, we recall from Sec. 15.12 that the motion of a body about a fixed point—or about its mass center—can be represented by the motion of a body cone rolling on a space cone. In the case of steady precession, the two cones are circular, since the angles $\gamma$ and $\theta - \gamma$ that the angular velocity $\mathbf{\omega}$ forms, respectively, with the axis of symmetry of the body and with the precession axis are constant. Two cases should be distinguished:

1. $I < I'$. This is the case of an elongated body, such as the space vehicle of Fig. 18.23. By (18.49) we have $\gamma < \theta$; the vector $\mathbf{\omega}$ lies inside the angle $\angle GZz$; the space cone and the body cone are tangent externally; the spin and the precession are both observed as counterclockwise from the positive $z$ axis. The precession is said to be direct.

2. $I > I'$. This is the case of a flattened body, such as the satellite of Fig. 18.24. By (18.49) we have $\gamma > \theta$; since the vector $\mathbf{\omega}$ must lie outside the angle $\angle GZz$, the vector $\mathbf{\psi}$ has a sense opposite to that of the $z$ axis; the space cone is inside the body cone; the precession and the spin have opposite senses. The precession is said to be retrograde.
SAMPLE PROBLEM 18.6

A space satellite of mass \( m \) is known to be dynamically equivalent to two thin disks of equal mass. The disks are of radius \( a = 800 \text{ mm} \) and are rigidly connected by a light rod of length \( 2a \). Initially the satellite is spinning freely about its axis of symmetry at the rate \( \omega_0 = 60 \text{ rpm} \). A meteorite, of mass \( m_0 = m/1000 \) and traveling with a velocity \( v_0 \) of 2000 m/s relative to the satellite, strikes the satellite and becomes embedded at \( C \). Determine (a) the angular velocity of the satellite immediately after impact, (b) the precession axis of the ensuing motion, (c) the rates of precession and spin of the ensuing motion.

SOLUTION

Moments of Inertia. We note that the axes shown are principal axes of inertia for the satellite and write

\[
I = I_x = m a^2, \quad I' = I_y = I_z = 2\left[\frac{1}{2}m a^2 + \frac{1}{2}m a^2\right] = \frac{3}{2}m a^2
\]

Principle of Impulse and Momentum. We consider the satellite and the meteorite as a single system. Since no external force acts on this system, the momenta before and after impact are equipollent. Taking moments about \( G \), we write

\[
2a m_0 v_0 k = H_G = -m_0 v_0 a i + I_0 k
\]

Angular Velocity after Impact. Substituting the values obtained for the components of \( H_G \) and for the moments of inertia into

\[
H_x = I_x \omega_x, \quad H_y = I_y \omega_y, \quad H_z = I_z \omega_z
\]

we write

\[
-m_0 v_0 a = I' \omega_x = \frac{3}{2}m a^2 \omega_x, \quad 0 = I' \omega_y, \quad I_0 = I_z \omega_z
\]

\[
\omega_x = \frac{-m_0 v_0 a}{\frac{3}{2}m a}, \quad \omega_y = 0, \quad \omega_z = \omega_0
\]

For the satellite considered we have \( \omega_0 = 60 \text{ rpm} = 6.283 \text{ rad/s} \), \( m_0/m = \frac{1}{1000} \), \( a = 0.800 \text{ m} \), and \( v_0 = 2000 \text{ m/s} \); we find

\[
\omega_x = -2 \text{ rad/s}, \quad \omega_y = 0, \quad \omega_z = 6.283 \text{ rad/s}
\]

\[
\omega = \sqrt{\omega_x^2 + \omega_z^2} = 6.594 \text{ rad/s}, \quad \tan \gamma = \frac{-\omega_x}{\omega_z} = +0.3183
\]

\[
\omega = 63.0 \text{ rpm}, \quad \gamma = 17.7^\circ
\]

Precession Axis. Since in free motion the direction of the angular momentum \( \mathbf{H}_G \) is fixed in space, the satellite will precess about this direction. The angle \( \theta \) formed by the precession axis and the \( z \) axis is

\[
\tan \theta = \frac{-H_x}{H_z} = \frac{m_0 v_0 a}{I_0} = \frac{2m_0 v_0 a}{m a \omega_0} = 0.796 \quad \theta = 38.5^\circ
\]

Rates of Precession and Spin. We sketch the space and body cones for the free motion of the satellite. Using the law of sines, we compute the rates of precession and spin.

\[
\frac{\omega}{\sin \theta} = \frac{\dot{\phi}}{\sin \gamma} = \frac{\dot{\psi}}{\sin (\theta - \gamma)}
\]

\[
\dot{\phi} = 30.8 \text{ rpm}, \quad \dot{\psi} = 35.9 \text{ rpm}
\]
SOLVING PROBLEMS
ON YOUR OWN

In this lesson we analyzed the motion of gyroscopes and of other axisymmetrical bodies with a fixed point O. In order to define the position of these bodies at any given instant, we introduced the three Eulerian angles $\phi$, $\theta$, and $\psi$ (Fig. 18.15), and noted that their time derivatives define, respectively, the rate of precession, the rate of nutation, and the rate of spin (Fig. 18.16). The problems you will encounter fall into one of the following categories.

1. Steady precession. This is the motion of a gyroscope or other axisymmetrical body with a fixed point located on its axis of symmetry, in which the angle $\theta$, the rate of precession $\dot{\phi}$, and the rate of spin $\dot{\psi}$ all remain constant.

   a. Using the rotating frame of reference $Oxyz$ shown in Fig. 18.17, which precesses with the body, but does not spin with it, we obtained the following expressions for the angular velocity $\omega$ of the body, its angular momentum $H_O$, and the angular velocity $\Omega$ of the frame $Oxyz$:

   \[
   \omega = -\dot{\phi} \sin \theta \mathbf{i} + \omega_z \mathbf{k} \tag{18.40}
   \]

   \[
   H_O = -I'\dot{\phi} \sin \theta \mathbf{i} + I \omega_z \mathbf{k} \tag{18.41}
   \]

   \[
   \Omega = -\phi \sin \theta \mathbf{i} + \phi \cos \theta \mathbf{k} \tag{18.42}
   \]

   where $I$ = moment of inertia of body about its axis of symmetry

   $I'$ = moment of inertia of body about a transverse axis through $O$

   $\omega_z = $ rectangular component of $\omega$ along z axis = $\psi + \phi \cos \theta$

   b. The sum of the moments about $O$ of the forces applied to the body is equal to the rate of change of its angular momentum, as expressed by Eq. (18.28). But, since $\theta$ and the rates of change $\dot{\phi}$ and $\dot{\psi}$ are constant, it follows from Eq. (18.41) that $H_O$ remains constant in magnitude and direction when viewed from the frame $Oxyz$. Thus, its rate of change is zero with respect to that frame and you can write

   \[
   \Sigma M_O = \Omega \times H_O \tag{18.43}
   \]

   where $\Omega$ and $H_O$ are defined, respectively by Eq. (18.42) and Eq. (18.41). The equation obtained shows that the moment resultant at $O$ of the forces applied to the body is perpendicular to both the axis of precession and the axis of spin (Fig. 18.18).

   c. Keep in mind that the method described applies, not only to gyroscopes, where the fixed point $O$ coincides with the mass center $G$, but also to any axisymmetrical body with a fixed point $O$ located on its axis of symmetry. This method, therefore, can be used to analyze the steady precession of a top on a rough floor.

   d. When an axisymmetrical body has no fixed point, but is in steady precession about its mass center $G$, you should draw a free-body-diagram equation showing that the system of the external forces exerted on the body (including the body’s weight) is equivalent to the vector $m\mathbf{a}$ applied at $G$ and the couple vector.
\( \mathbf{H}_C \). You can use Eqs. (18.40) through (18.42), replacing \( \mathbf{H}_O \) with \( \mathbf{H}_C \), and express the moment of the couple as

\[ \dot{\mathbf{H}}_C = \Omega \times \mathbf{H}_C \]

You can then use the free-body-diagram equation to write as many as six independent scalar equations.

2. Motion of an axisymmetrical body under no force, except its own weight. We have \( \Sigma \mathbf{M}_C = 0 \) and, thus, \( \dot{\mathbf{H}}_C = 0 \); it follows that the angular momentum \( \mathbf{H}_C \) is constant in magnitude and direction (Sec. 18.11). The body is in steady precession with the precession axis \( \mathbf{GZ} \) directed along \( \mathbf{H}_C \) (Fig. 18.20). Using the rotating frame \( \mathbf{Gxyz} \) and denoting by \( \gamma \) the angle that \( \mathbf{GZ} \) forms with the spin axis \( \mathbf{Gz} \) (Fig. 18.21), we obtained the following relation between \( \gamma \) and the angle \( \theta \) formed by the precession and spin axes:

\[ \tan \gamma = \frac{I}{I'} \tan \theta \]  

(18.49)

The precession is said to be direct if \( I < I' \) (Fig. 18.23) and retrograde if \( I > I' \) (Fig. 18.24).

a. In many of the problems dealing with the motion of an axisymmetrical body under no force, you will be asked to determine the precession axis and the rates of precession and spin of the body, knowing the magnitude of its angular velocity \( \omega \) and the angle \( \gamma \) that it forms with the axis of symmetry \( \mathbf{Gz} \) (Fig. 18.21). From Eq. (18.49) you will determine the angle \( \theta \) that the precession axis \( \mathbf{GZ} \) forms with \( \mathbf{Gz} \) and resolve \( \omega \) into its two oblique components, \( \phi \mathbf{k} \) and \( \psi \mathbf{k} \). Using the law of sines, you will then determine the rate of precession \( \dot{\phi} \) and the rate of spin \( \dot{\psi} \).

b. In other problems, the body will be subjected to a given impulse and you will first determine the resulting angular momentum \( \mathbf{H}_C \). Using Eqs. (18.10), you will calculate the rectangular components of the angular velocity \( \omega \), its magnitude \( \omega \), and the angle \( \gamma \) that it forms with the axis of symmetry. You will then determine the precession axis and the rates of precession and spin as described above [Sample Prob. 18.6].

3. General motion of an axisymmetric body with a fixed point \( \mathbf{O} \) located on its axis of symmetry, and subjected only to its own weight. This is a motion in which the angle \( \theta \) is allowed to vary. At any given instant you should take into account the rate of precession \( \dot{\phi} \), the rate of spin \( \dot{\psi} \), and the rate of nutation \( \dot{\theta} \), none of which will remain constant. An example of such a motion is the motion of a top, which is discussed in Probs. 18.139 and 18.140. The rotating frame of reference \( \mathbf{Oxyz} \) that you will use is still the one shown in Fig. 18.18, but this frame
will now rotate about the \( y \) axis at the rate \( \dot{\theta} \). Equations (18.40), (18.41), and (18.42), therefore, should be replaced by the following equations:

\[
\begin{align*}
\omega &= -\dot{\phi} \sin\theta \mathbf{i} + \dot{\theta} \mathbf{j} + (\dot{\psi} + \dot{\phi} \cos\theta) \mathbf{k} \quad (18.40') \\
H_O &= -I' \dot{\phi} \sin\theta \mathbf{i} + I' \dot{\theta} \mathbf{j} + I(\dot{\psi} + \dot{\phi} \cos\theta) \mathbf{k} \quad (18.41') \\
\Omega &= -\dot{\phi} \sin\theta \mathbf{i} + \dot{\theta} \mathbf{j} + \dot{\phi} \cos\theta \mathbf{k} \quad (18.42')
\end{align*}
\]

Since substituting these expressions into Eq. (18.44) would lead to nonlinear differential equations, it is preferable, whenever feasible, to apply the following conservation principles.

**a. Conservation of energy.** Denoting by \( c \) the distance between the fixed point \( O \) and the mass center \( G \) of the body, and by \( E \) the total energy, you will write

\[
T + V = E: \quad \frac{1}{2}(I'\omega^2 + I'\omega^2_y + I\omega^2_z) + mc \cos\theta = E
\]

and substitute for the components of \( \omega \) the expressions obtained in Eq. (18.40'). Note that \( c \) will be positive or negative, depending upon the position of \( G \) relative to \( O \). Also, \( c = 0 \) if \( G \) coincides with \( O \); the kinetic energy is then conserved.

**b. Conservation of the angular momentum about the axis of precession.** Since the support at \( O \) is located on the \( Z \) axis, and since the weight of the body and the \( Z \) axis are both vertical and, thus, parallel to each other, it follows that \( \Sigma M_Z = 0 \) and, thus, that \( H_Z \) remains constant. This can be expressed by writing that the scalar product \( \mathbf{K} \cdot \mathbf{H}_O \) is constant, where \( \mathbf{K} \) is the unit vector along the \( Z \) axis.

**c. Conservation of the angular momentum about the axis of spin.** Since the support at \( O \) and the center of gravity \( G \) are both located on the \( z \) axis, it follows that \( \Sigma M_z = 0 \) and, thus, that \( H_z \) remains constant. This is expressed by writing that the coefficient of the unit vector \( \mathbf{k} \) in Eq. (18.41') is constant. Note that this last conservation principle cannot be applied when the body is restrained from spinning about its axis of symmetry, but in that case the only variables are \( \theta \) and \( \phi \).
18.107 A solid aluminum sphere of radius 3 in. is welded to the end of a 6-in.-long rod \( AB \) of negligible mass which is supported by a ball-and-socket joint at \( A \). Knowing that the sphere is observed to precess about a vertical axis at the constant rate of 60 rpm in the sense indicated and that rod \( AB \) forms an angle \( \beta = 30^\circ \) with the vertical, determine the rate of spin of the sphere about line \( AB \).

18.108 A solid aluminum sphere of radius 3 in. is welded to the end of a 6-in.-long rod \( AB \) of negligible mass which is supported by a ball-and-socket joint at \( A \). Knowing that the sphere spins as shown about line \( AB \) at the rate of 700 rpm, determine the angle \( \beta \) for which the sphere will precess about a vertical axis at the constant rate of 60 rpm in the sense indicated.

18.109 A solid cone of height 9 in. with a circular base of radius 3 in, is supported by a ball-and-socket joint at \( A \). Knowing that the cone is observed to precess about the vertical axis \( AC \) at the constant rate of 40 rpm in the sense indicated and that its axis of symmetry \( AB \) forms an angle \( \beta = 40^\circ \) with \( AC \), determine the rate at which the cone spins about the axis \( AB \).

18.110 A solid cone of height 9 in. with a circular base of radius 3 in. is supported by a ball-and-socket joint at \( A \). Knowing that the cone is spinning about its axis of symmetry \( AB \) at the rate of 3000 rpm and that \( AB \) forms an angle \( \beta = 60^\circ \) with the vertical axis \( AC \), determine the two possible rates of steady precession of the cone about the axis \( AC \).

18.111 The 85-g top shown is supported at the fixed point \( O \). The radii of gyration of the top with respect to its axis of symmetry and with respect to a transverse axis through \( O \) are 21 mm and 45 mm, respectively. Knowing that \( c = 37.5 \) mm and that the rate of spin of the top about its axis of symmetry is 1800 rpm, determine the two possible rates of steady precession corresponding to \( \theta = 30^\circ \).

18.112 The top shown is supported at the fixed point \( O \) and its moments of inertia about its axis of symmetry and about a transverse axis through \( O \) are denoted, respectively, by \( I \) and \( I' \). (a) Show that the condition for steady precession of the top is
\[
(I_\omega - I' \hat{\phi} \cos \theta) \hat{\phi} = We
\]
where \( \hat{\phi} \) is the rate of precession and \( \omega_\tau \) is the rectangular component of the angular velocity along the axis of symmetry of the top.

(b) Show that if the rate of spin \( \psi \) of the top is very large compared with its rate of precession \( \hat{\phi} \), the condition for steady precession is \( I \hat{\phi} \approx We \).

(c) Determine the percentage error introduced when this last relation is used to approximate the slower of the two rates of precession obtained for the top of Prob. 18.111.
18.113 A solid cube of side \( c = 80 \text{ mm} \) is attached as shown to cord \( AB \). It is observed to spin at the rate \( \dot{\psi} = 40 \text{ rad/s} \) about its diagonal \( BC \) and to precess at the constant rate \( \dot{\phi} = 5 \text{ rad/s} \) about the vertical axis \( AD \). Knowing that \( \beta = 30^\circ \), determine the angle \( \theta \) that the diagonal \( BC \) forms with the vertical. (Hint: The moment of inertia of a cube about an axis through its center is independent of the orientation of that axis.)

18.114 A solid cube of side \( c = 120 \text{ mm} \) is attached as shown to a cord \( AB \) of length 240 mm. The cube spins about its diagonal \( BC \) and precesses about the vertical axis \( AD \). Knowing that \( \theta = 25^\circ \) and \( \beta = 40^\circ \), determine (a) the rate of spin of the cube, (b) its rate of precession. (See hint of Prob. 18.113.)

18.115 A solid sphere of radius \( c = 3 \text{ in.} \) is attached as shown to cord \( AB \). The sphere is observed to precess at the constant rate \( \dot{\phi} = 6 \text{ rad/s} \) about the vertical axis \( AD \). Knowing that \( \beta = 40^\circ \), determine the angle \( \theta \) that the diameter \( BC \) forms with the vertical when the sphere (a) has no spin, (b) spins about its diameter \( BC \) at the rate \( \dot{\psi} = 50 \text{ rad/s} \), (c) spins about \( BC \) at the rate \( \dot{\psi} = -50 \text{ rad/s} \).

18.116 A solid sphere of radius \( c = 3 \text{ in.} \) is attached as shown to a cord \( AB \) of length 15 in. The sphere spins about its diameter \( BC \) and precesses about the vertical axis \( AD \). Knowing that \( \theta = 20^\circ \) and \( \beta = 35^\circ \), determine (a) the rate of spin of the sphere, (b) its rate of precession.

18.117 If the earth were a sphere, the gravitational attraction of the sun, moon, and planets would at all times be equivalent to a single force \( \mathbf{R} \) acting at the mass center of the earth. However, the earth is actually an oblate spheroid and the gravitational system acting on the earth is equivalent to a force \( \mathbf{R} \) and a couple \( \mathbf{M} \). Knowing that the effect of the couple \( \mathbf{M} \) is to cause the axis of the earth to precess about the axis \( GA \) at the rate of one revolution in 25 800 years, determine the average magnitude of the couple \( \mathbf{M} \) applied to the earth. Assume that the average density of the earth is 5.51 \( \text{g/cm}^3 \), that the average radius of the earth is 6370 km, and that \( I = \frac{2}{5}mr^2 \). (Note: This forced precession is known as the precession of the equinoxes and is not to be confused with the free precession discussed in Prob. 18.123.)
18.118 A high-speed photographic record shows that a certain projectile was fired with a horizontal velocity $\mathbf{v}$ of 600 m/s and with its axis of symmetry forming an angle $\beta = 3^\circ$ with the horizontal. The rate of spin $\psi$ of the projectile was 6000 rpm, and the atmospheric drag was equivalent to a force $\mathbf{D}$ of 120 N acting at the center of pressure $C_p$ located at a distance $c = 150$ mm from $G$. (a) Knowing that the projectile has a mass of 20 kg and a radius of gyration of 50 mm with respect to its axis of symmetry, determine its approximate rate of steady precession. (b) If it is further known that the radius of gyration of the projectile with respect to a transverse axis through $G$ is 200 mm, determine the exact values of the two possible rates of precession.

18.119 Show that for an axisymmetrical body under no force, the rates of precession and spin can be expressed, respectively, as

$$\dot{\phi} = \frac{H_G}{I'}$$

and

$$\dot{\psi} = \frac{H_G \cos \theta (I' - I)}{I'}$$

where $H_G$ is the constant value of the angular momentum of the body.

18.120 (a) Show that for an axisymmetrical body under no force, the rate of precession can be expressed as

$$\dot{\phi} = \frac{I \omega_z}{I' \cos \theta}$$

where $\omega_z$ is the rectangular component of $\mathbf{\omega}$ along the axis of symmetry of the body. (b) Use this result to check that the condition (18.44) for steady precession is satisfied by an axisymmetrical body under no force.

18.121 Show that the angular velocity vector $\mathbf{\omega}$ of an axisymmetrical body under no force is observed from the body itself to rotate about the axis of symmetry at the constant rate

$$\eta = \frac{(I' - I)}{I'} \omega_z$$

where $\omega_z$ is the rectangular component of $\mathbf{\omega}$ along the axis of symmetry of the body.

18.122 For an axisymmetrical body under no force, prove (a) that the rate of retrograde precession can never be less than twice the rate of spin of the body about its axis of symmetry, (b) that in Fig. 18.24 the axis of symmetry of the body can never lie within the space cone.

18.123 Using the relation given in Prob. 18.121, determine the period of precession of the north pole of the earth about the axis of symmetry of the earth. The earth may be approximated by an oblate spheroid of axial moment of inertia $I$ and of transverse moment of inertia $I' = 0.9967I$. (Note: Actual observations show a period of precession of the north pole of about 432.5 mean solar days; the difference between the observed and computed periods is due to the fact that the earth is not a perfectly rigid body. The free precession considered here should not be confused with the much slower precession of the equinoxes, which is a forced precession. See Prob. 18.117.)
The angular velocity vector of a football which has just been kicked is horizontal, and its axis of symmetry $OC$ is oriented as shown. Knowing that the magnitude of the angular velocity is 200 rpm and that the ratio of the axis and transverse moments of inertia is $I/I' = 2$, determine (a) the orientation of the axis of precession $OA$, (b) the rates of precession and spin.

A 2500-kg satellite is 2.4 m high and has octagonal bases of sides 1.2 m. The coordinate axes shown are the principal centroidal axes of inertia of the satellite, and its radii of gyration are $k_x = k_z = 0.90$ m and $k_y = 0.98$ m. The satellite is equipped with a main 500-N thruster $E$ and four 20-N thrusters $A$, $B$, $C$, and $D$ which can expel fuel in the positive $y$ direction. The satellite is spinning at the rate of 36 rev/h about its axis of symmetry $Gy$, which maintains a fixed direction in space, when thrusters $A$ and $B$ are activated for 2 s. Determine (a) the precession axis of the satellite, (b) its rate of precession, (c) its rate of spin.

A 800-lb geostationary satellite is spinning with an angular velocity $\omega_0 = (1.5 \text{ rad/s})\hat{j}$ when it is hit at $B$ by a 6-oz meteorite traveling with a velocity $v_0 = -(1600 \text{ ft/s})\hat{i} + (1300 \text{ ft/s})\hat{j} + (4000 \text{ ft/s})\hat{k}$ relative to the satellite. Knowing that $b = 20$ in. and that the radii of gyration of the satellite are $k_x = k_y = 28.8$ in. and $k_z = 32.4$ in., determine the precession axis and the rates of precession and spin of the satellite after the impact.

A coin is tossed into the air. It is observed to spin at the rate of 600 rpm about an axis $GC$ perpendicular to the coin and to precess about the vertical direction $GD$. Knowing that $GC$ forms an angle of 15° with $GD$, determine (a) the angle that the angular velocity $\omega$ of the coin forms with $GD$, (b) the rate of precession of the coin about $GD$. 
18.130 Solve Sample Prob. 18.6, assuming that the meteorite strikes the satellite at C with a velocity \( v_0 = (2000 \text{ m/s}) \).

18.131 A homogeneous disk of mass \( m \) is connected at A and B to a fork-ended shaft of negligible mass which is supported by a bearing at C. The disk is free to rotate about its horizontal diameter AB and the shaft is free to rotate about a vertical axis through C. Initially the disk lies in a vertical plane (\( \theta_0 = 90^\circ \)) and the shaft has an angular velocity \( \phi_0 = 8 \text{ rad/s} \). If the disk is slightly disturbed, determine for the ensuing motion (a) the minimum value of \( \phi \), (b) the maximum value of \( \theta \).

18.132 The slender homogeneous rod AB of mass \( m \) and length \( L \) is free to rotate about a horizontal axle through its mass center G. The axle is supported by a frame of negligible mass which is free to rotate about the vertical CD. Knowing that, initially, \( \theta = \theta_0 \), \( \dot{\theta} = 0 \), and \( \phi = \phi_0 \), show that the rod will oscillate about the horizontal axle and determine (a) the range of values of angle \( \theta \) during this motion, (b) the maximum value of \( \dot{\theta} \), (c) the minimum value of \( \dot{\phi} \).

18.133 A homogeneous rectangular plate of mass \( m \) and sides \( c \) and \( 2c \) is held at A and B by a fork-ended shaft of negligible mass which is supported by a bearing at C. The plate is free to rotate about AB, and the shaft is free to rotate about a horizontal axis through C. Knowing that, initially, \( \theta_0 = 30^\circ \), \( \dot{\theta}_0 = 0 \), and \( \dot{\phi}_0 = 6 \text{ rad/s} \), determine for the ensuing motion (a) the range of values of \( \theta \), (b) the minimum value of \( \dot{\theta} \), (c) the maximum value of \( \dot{\phi} \).

18.134 A homogeneous rectangular plate of mass \( m \) and sides \( c \) and \( 2c \) is held at A and B by a fork-ended shaft of negligible mass which is supported by a bearing at C. The plate is free to rotate about AB, and the shaft is free to rotate about a horizontal axis through C. Initially the plate lies in the plane of the fork (\( \theta_0 = 0 \)) and the shaft has an angular velocity \( \phi_0 = 6 \text{ rad/s} \). If the plate is slightly disturbed, determine for the ensuing motion (a) the minimum value of \( \phi \), (b) the maximum value of \( \theta \).
A homogeneous disk of radius 180 mm is welded to a rod AG of length 360 mm and of negligible mass which is connected by a clevis to a vertical shaft AB. The rod and disk can rotate freely about a horizontal axis AC, and shaft AB can rotate freely about a vertical axis. Initially rod AG is horizontal \( \theta_0 = 90^\circ \) and has no angular velocity about AC. Knowing that the maximum value \( \dot{\phi}_m \) of the angular velocity of shaft AB in the ensuing motion is twice its initial value \( \dot{\phi}_0 \), determine \((a)\) the minimum value of \( \theta \), \((b)\) the initial angular velocity \( \dot{\phi}_0 \) of shaft AB.

A homogeneous disk of radius 180 mm is welded to a rod AG of length 360 mm and of negligible mass which is connected by a clevis to a vertical shaft AB. The rod and disk can rotate freely about a horizontal axis AC, and shaft AB can rotate freely about a vertical axis. Initially rod AG is horizontal \( \theta_0 = 90^\circ \) and has no angular velocity about AC. Knowing that the smallest value of \( \theta \) in the ensuing motion is 30°, determine \((a)\) the initial angular velocity of shaft AB, \((b)\) its maximum angular velocity.

A homogeneous disk of radius 180 mm is welded to a rod AG of length 360 mm and of negligible mass which is supported by a ball and socket at A. The disk is released with a rate of spin \( \dot{\psi}_0 = 50 \text{ rad/s} \), with zero rates of precession and nutation, and with rod AG horizontal \( \theta_0 = 90^\circ \). Determine \((a)\) the smallest value of \( \theta \) in the ensuing motion, \((b)\) the rates of precession and spin as the disk passes through its lowest position.

A homogeneous disk of radius 180 mm is welded to a rod AG of length 360 mm and of negligible mass which is supported by a ball and socket at A. The disk is released with a rate of spin \( \dot{\psi}_0 \) counterclockwise as seen from A, with zero rates of precession and nutation, and with rod AG horizontal \( \theta_0 = 90^\circ \). Knowing that the smallest value of \( \theta \) in the ensuing motion is 30°, determine \((a)\) the rate of spin \( \dot{\psi}_0 \) of the disk in its initial position, \((b)\) the rates of precession and spin as the disk passes through its lowest position.
The top shown is supported at the fixed point $O$. Denoting by $\phi$, $\theta$, and $\psi$ the Eulerian angles defining the position of the top with respect to a fixed frame of reference, consider the general motion of the top in which all Eulerian angles vary.

(a) Observing that $\Sigma M_\phi = 0$ and $\Sigma M_\psi = 0$, and denoting by $I$ and $I'$, respectively, the moments of inertia of the top about its axis of symmetry and about a transverse axis through $O$, derive the two first-order differential equations of motion

$$
I' \phi \sin^2 \theta + I(\psi + \phi \cos \theta) \cos \theta = \alpha
$$

$$
I(\dot{\psi} + \dot{\phi} \cos \theta) = \beta
$$

where $\alpha$ and $\beta$ are constants depending upon the initial conditions. These equations express that the angular momentum of the top is conserved about both the $Z$ and $z$ axes, i.e., that the rectangular component of $\mathbf{H}_O$ along each of these axes is constant.

(b) Use Eqs. (1) and (2) to show that the rectangular component $v_z$ of the angular velocity of the top is constant and that the rate of precession $\dot{\phi}$ depends upon the value of the angle of nutation $\theta$.

Applying the principle of conservation of energy, derive a third differential equation for the general motion of the top of Prob. 18.139.

(b) Eliminating the derivatives $\phi$ and $\psi$ from the equation obtained and from the two equations of Prob. 18.139, show that the rate of nutation $\dot{\theta}$ is defined by the differential equation

$$
\dot{\theta}^2 = f(\theta),
$$

where

$$
f(\theta) = \frac{1}{I} \left( 2E - \frac{\beta^2}{I} - 2mgc \cos \theta \right) - \left( \frac{\alpha - \beta \cos \theta}{I' \sin \theta} \right)^2
$$

(c) Further show, by introducing the auxiliary variable $x = \cos \theta$, that the maximum and minimum values of $\theta$ can be obtained by solving for $x$ the cubic equation

$$
\left( 2E - \frac{\beta^2}{I} - 2mgc \right)(1 - x^2) - \frac{1}{I} (\alpha - \beta x)^2 = 0
$$

A homogeneous sphere of mass $m$ and radius $a$ is welded to a rod $AB$ of negligible mass, which is held by a ball-and-socket support at $A$. The sphere is released in the position $\beta = 0$ with a rate of precession $\dot{\phi}_0 = \sqrt{17g/11\mu}$ with no spin or nutation. Determine the largest value of $\beta$ in the ensuing motion.

A homogeneous sphere of mass $m$ and radius $a$ is welded to a rod $AB$ of negligible mass, which is held by a ball-and-socket support at $A$. The sphere is released in the position $\beta = 0$ with a rate of precession $\dot{\phi} = \phi_0$ with no spin or nutation. Knowing that the largest value of $\beta$ in the ensuing motion is $30^\circ$, determine (a) the rate of precession $\phi_0$ of the sphere in its initial position, (b) the rates of precession and spin when $\beta = 30^\circ$. 
**18.143** Consider a rigid body of arbitrary shape which is attached at its mass center $O$ and subjected to no force other than its weight and the reaction of the support at $O$.

(a) Prove that the angular momentum $H_O$ of the body about the fixed point $O$ is constant in magnitude and direction, that the kinetic energy $T$ of the body is constant, and that the projection along $H_O$ of the angular velocity $\omega$ of the body is constant.

(b) Show that the tip of the vector $\omega$ describes a curve on a fixed plane in space (called the invariable plane), which is perpendicular to $H_O$ and at a distance $2T/H_O$ from $O$.

(c) Show that with respect to a frame of reference attached to the body and coinciding with its principal axes of inertia, the tip of the vector $\omega$ appears to describe a curve on an ellipsoid of equation

$$I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 = 2T = \text{constant}$$

The ellipsoid (called the Poinsot ellipsoid) is rigidly attached to the body and is of the same shape as the ellipsoid of inertia, but of a different size.

**18.144** Referring to Prob. 18.143, (a) prove that the Poinsot ellipsoid is tangent to the invariable plane, (b) show that the motion of the rigid body must be such that the Poinsot ellipsoid appears to roll on the invariable plane. [Hint: In part $a$, show that the normal to the Poinsot ellipsoid at the tip of $\omega$ is parallel to $H_O$. It is recalled that the direction of the normal to a surface of equation $F(x, y, z) = \text{constant}$ at a point $P$ is the same as that of $\nabla F$ at point $P$.]

**18.145** Using the results obtained in Probs. 18.143 and 18.144, show that for an axisymmetrical body attached at its mass center $O$ and under no force other than its weight and the reaction at $O$, the Poinsot ellipsoid is an ellipsoid of revolution and the space and body cones are both circular and are tangent to each other. Further show that (a) the two cones are tangent externally, and the precession is direct, when $I < I'$, where $I$ and $I'$ denote, respectively, the axial and transverse moment of inertia of the body, (b) the space cone is inside the body cone, and the precession is retrograde, when $I > I'$.

**18.146** Refer to Probs. 18.143 and 18.144.

(a) Show that the curve (called polhode) described by the tip of the vector $\omega$ with respect to a frame of reference coinciding with the principal axes of inertia of the rigid body is defined by the equations

$$I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 = 2T = \text{constant} \quad (1)$$

$$I_x^2\omega_x^2 + I_y^2\omega_y^2 + I_z^2\omega_z^2 = H_O^2 = \text{constant} \quad (2)$$

and that this curve can, therefore, be obtained by intersecting the Poinsot ellipsoid with the ellipsoid defined by Eq. (2).

(b) Further show, assuming $I_x > I_y > I_z$, that the polhodes obtained for various values of $H_O$ have the shapes indicated in the figure.

(c) Using the result obtained in part $b$, show that a rigid body under no force can rotate about a fixed centroidal axis if, and only if, that axis coincides with one of the principal axes of inertia of the body, and that the motion will be stable if the axis of rotation coincides with the major or minor axis of the Poinsot ellipsoid ($z$ or $x$ axis in the figure) and unstable if it coincides with the intermediate axis ($y$ axis).
This chapter was devoted to the kinetic analysis of the motion of rigid bodies in three dimensions.

We first noted [Sec. 18.1] that the two fundamental equations derived in Chap. 14 for the motion of a system of particles

\[ \Sigma F = m\ddot{a} \] \hspace{1cm} (18.1)
\[ \Sigma M_G = \dot{H}_G \] \hspace{1cm} (18.2)

provide the foundation of our analysis, just as they did in Chap. 16 in the case of the plane motion of rigid bodies. The computation of the angular momentum \( H_G \) of the body and of its derivative \( \dot{H}_G \), however, are now considerably more involved.

In Sec. 18.2, we saw that the rectangular components of the angular momentum \( H_G \) of a rigid body can be expressed as follows in terms of the components of its angular velocity \( \omega \) and of its centroidal moments and products of inertia:

\[
H_x = \dot{I}_x \omega_x - \dot{I}_{xy} \omega_y - \dot{I}_{xz} \omega_z \\
H_y = -\dot{I}_{yx} \omega_x + \dot{I}_y \omega_y - \dot{I}_{yz} \omega_z \\
H_z = -\dot{I}_{zx} \omega_x + \dot{I}_{zy} \omega_y + \dot{I}_z \omega_z
\] \hspace{1cm} (18.7)

If principal axes of inertia \( Gx'y'z' \) are used, these relations reduce to

\[
H_{x'} = \dot{I}_{x'} \omega_{x'} \\
H_{y'} = \dot{I}_{y'} \omega_{y'} \\
H_{z'} = \dot{I}_{z'} \omega_{z'} 
\] \hspace{1cm} (18.10)

We observed that, in general, the angular momentum \( H_G \) and the angular velocity \( \omega \) do not have the same direction (Fig. 18.25). They will, however, have the same direction if \( \omega \) is directed along one of the principal axes of inertia of the body.
Recalling that the system of the momenta of the particles forming a rigid body can be reduced to the vector $m\vec{v}$ attached at $G$ and the couple $\mathbf{H}_G$ (Fig. 18.26), we noted that, once the linear momentum $m\vec{v}$ and the angular momentum $\mathbf{H}_G$ of a rigid body have been determined, the angular momentum $\mathbf{H}_O$ of the body about any given point $O$ can be obtained by writing

$$\mathbf{H}_O = \vec{r} \times m\vec{v} + \mathbf{H}_G \quad (18.11)$$

In the particular case of a rigid body constrained to rotate about a fixed point $O$, the components of the angular momentum $\mathbf{H}_O$ of the body about $O$ can be obtained directly from the components of its angular velocity and from its moments and products of inertia with respect to axes through $O$. We wrote

$$
\begin{align*}
H_x &= +I_x\omega_x - I_y\omega_y - I_z\omega_z \\
H_y &= -I_y\omega_x + I_y\omega_y - I_z\omega_z \\
H_z &= -I_z\omega_x - I_y\omega_y + I_z\omega_z
\end{align*} \quad (18.13)
$$

The principle of impulse and momentum for a rigid body in three-dimensional motion [Sec. 18.3] is expressed by the same fundamental formula that was used in Chap. 17 for a rigid body in plane motion,

$$\text{Syst Momenta}_1 + \text{Syst Ext Imp}_{1\rightarrow 2} = \text{Syst Momenta}_2 \quad (17.4)$$

but the systems of the initial and final momenta should now be represented as shown in Fig. 18.26, and $\mathbf{H}_G$ should be computed from the relations (18.7) or (18.10) [Sample Probs. 18.1 and 18.2].

The kinetic energy of a rigid body in three-dimensional motion can be divided into two parts [Sec. 18.4], one associated with the motion of its mass center $G$ and the other with its motion about $G$. Using principal centroidal axes $x', y', z'$, we wrote

$$T = \frac{1}{2}mv^2 + \frac{1}{2}(\bar{I}_x\omega_x^2 + \bar{I}_y\omega_y^2 + \bar{I}_z\omega_z^2) \quad (18.17)$$

where $\bar{v}$ = velocity of mass center

$\omega$ = angular velocity

$m$ = mass of rigid body

$\bar{I}_x, \bar{I}_y, \bar{I}_z$ = principal centroidal moments of inertia
We also noted that, in the case of a rigid body constrained to rotate about a fixed point O, the kinetic energy of the body can be expressed as

\[ T = \frac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2) \]  

(18.20)

where the \( x' \), \( y' \), and \( z' \) axes are the principal axes of inertia of the body at \( O \). The results obtained in Sec. 18.4 make it possible to extend to the three-dimensional motion of a rigid body the application of the principle of work and energy and of the principle of conservation of energy.

The second part of the chapter was devoted to the application of the fundamental equations

\[ \Sigma \mathbf{F} = m\ddot{\mathbf{a}} \]  

(18.1)

\[ \Sigma \mathbf{M}_G = \dot{\mathbf{H}}_G \]  

(18.2)

to the motion of a rigid body in three dimensions. We first recalled [Sec. 18.5] that \( \mathbf{H}_G \) represents the angular momentum of the body relative to a centroidal frame \( GX'Y'Z' \) of fixed orientation (Fig. 18.27)

and that \( \dot{\mathbf{H}}_G \) in Eq. (18.2) represents the rate of change of \( \mathbf{H}_G \) with respect to that frame. We noted that, as the body rotates, its moments and products of inertia with respect to the frame \( GX'Y'Z' \) change continually. Therefore, it is more convenient to use a rotating frame \( Gxyz \) when resolving \( \dot{\mathbf{w}} \) into components and computing the moments and products of inertia that will be used to determine \( \mathbf{H}_G \) from Eqs. (18.7) or (18.10). However, since \( \dot{\mathbf{H}}_G \) in Eq. (18.2) represents the rate of change of \( \mathbf{H}_G \) with respect to the frame \( GX'Y'Z' \) of fixed orientation, we must use the method of Sec. 15.10 to determine its value. Recalling Eq. (15.31), we wrote

\[ \dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \Omega \times \mathbf{H}_G \]  

(18.22)

where \( \mathbf{H}_G \) = angular momentum of body with respect to frame \( GX'Y'Z' \) of fixed orientation

\( (\dot{\mathbf{H}}_G)_{Gxyz} \) = rate of change of \( \mathbf{H}_G \) with respect to rotating frame \( Gxyz \), to be computed from relations (18.7)

\( \Omega \) = angular velocity of the rotating frame \( Gxyz \)
Substituting for $\dot{H}_G$ from (18.22) into (18.2), we obtained

$$\sum M_G = (\dot{H}_G)_{Oxyz} + \Omega \times H_G \quad (18.23)$$

If the rotating frame is actually attached to the body, its angular velocity $\Omega$ is identically equal to the angular velocity $\omega$ of the body. There are many applications, however, where it is advantageous to use a frame of reference which is not attached to the body but rotates in an independent manner [Sample Prob. 18.5].

Setting $\Omega = \omega$ in Eq. (18.23), using principal axes, and writing this equation in scalar form, we obtained Euler's equations of motion [Sec. 18.6]. A discussion of the solution of these equations and of the scalar equations corresponding to Eq. (18.1) led us to extend d'Alembert's principle to the three-dimensional motion of a rigid body and to conclude that the system of the external forces acting on the rigid body is not only equipollent, but actually equivalent to the effective forces of the body represented by the vector $m\overrightarrow{a}$ and the couple $\dot{H}_G$ (Fig. 18.28). Problems involving the three-dimensional motion of a rigid body can be solved by considering the free-body-diagram equation represented in Fig. 18.28 and writing appropriate scalar equations relating the components or moments of the external and effective forces [Sample Probs. 18.3 and 18.5].

**Rigid body with a fixed point**

In the case of a rigid body constrained to rotate about a fixed point $O$, an alternative method of solution, involving the moments of the forces and the rate of change of the angular momentum about point $O$, can be used. We wrote [Sec. 18.7]:

$$\sum M_O = (\dot{H}_O)_{Oxyz} + \Omega \times H_O \quad (18.28)$$

where $\sum M_O =$ sum of moments about $O$ of forces applied to rigid body

$H_O =$ angular momentum of body with respect to fixed frame $OXYZ$

$(\dot{H}_O)_{Oxyz} =$ rate of change of $H_O$ with respect to rotating frame $Oxyz$, to be computed from relations (18.13)

$\Omega =$ angular velocity of rotating frame $Oxyz$

This approach can be used to solve certain problems involving the rotation of a rigid body about a fixed axis [Sec. 18.8], for example, an unbalanced rotating shaft [Sample Prob. 18.4].
In the last part of the chapter, we considered the motion of gyroscopes and other axisymmetrical bodies. Introducing the Eulerian angles \( \phi, \theta, \) and \( \psi \) to define the position of a gyroscope (Fig. 18.29), we observed that their derivatives \( \dot{\phi}, \dot{\theta}, \) and \( \dot{\psi} \) represent, respectively, the rates of precession, nutation, and spin of the gyroscope [Sec. 18.9]. Expressing the angular velocity \( \omega \) in terms of these derivatives, we wrote

\[
\omega = -\dot{\phi} \sin \theta \mathbf{i} + \dot{\theta} \mathbf{j} + (\dot{\psi} + \dot{\phi} \cos \theta) \mathbf{k} \quad (18.35)
\]

where the unit vectors are associated with a frame \( Oxyz \) attached to the inner gimbal of the gyroscope (Fig. 18.30) and rotate, therefore, with the angular velocity

\[
\Omega = -\dot{\phi} \sin \theta \mathbf{i} + \dot{\theta} \mathbf{j} + \dot{\phi} \cos \theta \mathbf{k} \quad (18.38)
\]

Denoting by \( I \) the moment of inertia of the gyroscope with respect to its spin axis \( z \) and by \( I' \) its moment of inertia with respect to a transverse axis through \( O \), we wrote

\[
\mathbf{H}_O = -I' \dot{\phi} \sin \theta \mathbf{i} + I' \dot{\theta} \mathbf{j} + I(\dot{\psi} + \dot{\phi} \cos \theta) \mathbf{k} \quad (18.36)
\]

Substituting for \( \mathbf{H}_O \) and \( \Omega \) into Eq. (18.28) led us to the differential equations defining the motion of the gyroscope.

In the particular case of the \textit{steady precession} of a gyroscope [Sec. 18.10], the angle \( \theta \), the rate of precession \( \dot{\phi} \), and the rate of spin \( \dot{\psi} \) remain constant. We saw that such a motion is possible only if the moments of the external forces about \( O \) satisfy the relation

\[
\Sigma \mathbf{M}_O = (I \omega_z - I' \dot{\phi} \cos \theta) \dot{\phi} \sin \theta \mathbf{j} \quad (18.44)
\]

i.e., if the external forces reduce to a couple of moment equal to the right-hand member of Eq. (18.44) and applied \textit{about an axis perpendicular to the precession axis and to the spin axis} (Fig. 18.31). The chapter ended with a discussion of the motion of an axisymmetrical body spinning and precessing \textit{under no force} [Sec. 18.11; Sample Prob. 18.6].
18.147 A homogeneous disk of mass \( m = 5 \text{ kg} \) rotates at the constant rate \( \omega_1 = 8 \text{ rad/s} \) with respect to the bent axle ABC, which itself rotates at the constant rate \( \omega_2 = 5 \text{ rad/s} \) about the y axis. Determine the angular momentum \( \mathbf{H}_C \) of the disk about its center C.

18.148 Two L-shaped arms, each weighing 5 lb, are welded to the one-third points of the 24-in. shaft AB. Knowing that shaft AB rotates at the constant rate \( \omega = 180 \text{ rpm} \), determine (a) the angular momentum \( \mathbf{H}_A \) of the body about A, (b) the angle that \( \mathbf{H}_A \) forms with the shaft.

18.149 A uniform rod of mass \( m \) and length 5a is bent into the shape shown and is suspended from a wire attached at B. Knowing that the rod is hit at C in the negative z direction and denoting the corresponding impulse by \( -(F \Delta t)\mathbf{k} \), determine immediately after the impact (a) the angular velocity of the rod, (b) the velocity of its mass center G.

18.150 A homogeneous disk of radius \( a \) and mass \( m \) supported by a ball-and-socket joint at A is rotating about its vertical diameter with a constant angular velocity \( \omega = \omega_0 \mathbf{j} \) when an obstruction is suddenly introduced at B. Assuming the impact to be perfectly plastic \( (e = 0) \), determine immediately after the impact (a) the angular velocity of the disk, (b) the velocity of its mass center G.
18.151 Determine the kinetic energy lost when the disk of Prob. 18.150 hits the obstruction at B.

18.152 Each of the two triangular plates shown has a mass of 5 kg and is welded to a vertical shaft AB. Knowing that the assembly rotates at the constant rate $\omega = 8$ rad/s, determine the dynamic reactions at A and B.

18.153 A 2.4-kg piece of sheet steel with dimensions $160 \times 640$ mm was bent to form the component shown. The component is at rest ($\omega = 0$) when a couple $M_0 = (0.8 \text{ N} \cdot \text{m})$ is applied to it. Determine (a) the angular acceleration of the component, (b) the dynamic reactions at A and B immediately after the couple is applied.

18.154 A thin ring of 3-in. radius is attached by a collar at A to a vertical shaft which rotates at the constant rate $\omega$. Determine (a) the constant angle $\beta$ that the plane of the ring forms with the vertical when $\omega = 12$ rad/s, (b) the maximum value of $\omega$ for which the ring will remain vertical ($\beta = 0$).

18.155 A thin disk of weight $W = 8 \text{ lb}$ rotates with an angular velocity $\omega_2$ with respect to arm OA, which itself rotates with an angular velocity $\omega_1$ about the $y$ axis. Determine (a) the couple $M_1 \mathbf{j}$ which should be applied to arm OA to give it an angular acceleration $\alpha_1 = (6 \text{ rad/s}^2) \mathbf{j}$ with $\omega_1 = 4 \text{ rad/s}$, knowing that the disk rotates at the constant rate $\omega_2 = 12 \text{ rad/s}$, (b) the force-couple system representing the dynamic reaction at O at that instant. Assume that arm OA has negligible mass.
18.156 An experimental Fresnel-lens solar-energy concentrator can rotate about the horizontal axis $AB$ which passes through its mass center $G$. It is supported at $A$ and $B$ by a steel framework which can rotate about the vertical $y$ axis. The concentrator has a mass of 30 Mg, a radius of gyration of 12 m about its axis of symmetry $CD$, and a radius of gyration of 10 m about any transverse axis through $G$. Knowing that the angular velocities $\omega_1$ and $\omega_2$ have constant magnitudes equal to 0.20 rad/s and 0.25 rad/s, respectively, determine for the position $u = 60^\circ$ 

(a) the forces exerted on the concentrator at $A$ and $B$, 

(b) the couple $M_k$ applied to the concentrator at that instant.

18.157 A 2-kg disk of 150-mm diameter is attached to the end of a rod $AB$ of negligible mass which is supported by a ball-and-socket joint at $A$. If the disk is observed to precess about the vertical in the sense indicated at a constant rate of 36 rpm, determine the rate of spin $\dot{\psi}$ of the disk about $AB$.

18.158 The essential features of the gyrocompass are shown. The rotor spins at the rate $\psi$ about an axis mounted in a single gimbal, which may rotate freely about the vertical axis $AB$. The angle formed by the axis of the rotor and the plane of the meridian is denoted by $\theta$, and the latitude of the position on the earth is denoted by $\lambda$. We note that line $OC$ is parallel to the axis of the earth, and we denote by $\omega_c$ the angular velocity of the earth about its axis.

(a) Show that the equations of motion of the gyrocompass are

$$I' \ddot{\theta} + I \omega_c \cos \lambda \sin \theta = I' \omega_c^2 \cos^2 \lambda \sin \theta \cos \theta = 0$$

$$I \omega_z = 0$$

where $\omega_c$ is the rectangular component of the total angular velocity $\omega$ along the axis of the rotor, and $I$ and $I'$ are the moments of inertia of the rotor with respect to its axis of symmetry and a transverse axis through $O$, respectively.

(b) Neglecting the term containing $\omega_c^2$, show that for small values of $\theta$, we have

$$\ddot{\theta} + \frac{I \omega_c \cos \lambda}{I'} \dot{\theta} = 0$$

and that the axis of the gyrocompass oscillates about the north-south direction.
**COMPUTER PROBLEMS**

**18.C1** A wire of uniform cross section weighing $\frac{5}{8}$ oz/ft is used to form the wire figure shown, which is suspended from cord $AD$. An impulse $F \Delta t = (0.5 \text{ lb} \cdot \text{s})j$ is applied to the wire figure at point $E$. Use computational software to calculate and plot immediately after the impact, for values of $\theta$ from 0 to 180°, (a) the velocity of the mass center of the wire figure, (b) the angular velocity of the figure.

![Diagram of wire figure](image)

**Fig. P18.C1**

**18.C2** A 2500-kg probe in orbit about the moon is 2.4 m high and has octagonal bases of sides 1.2 m. The coordinate axes shown are the principal centroidal axes of inertia of the probe, and its radii of gyration are $k_x = 0.98 \text{ m}$, $k_y = 1.06 \text{ m}$, and $k_z = 1.02 \text{ m}$. The probe is equipped with a main 500-N thruster $E$ and four 20-N thrusters $A$, $B$, $C$, and $D$ that can expel fuel in the positive $y$ direction. The probe has an angular velocity $\omega = \omega_x \hat{i} + \omega_y \hat{k}$ when two of the 20-N thrusters are used to reduce the angular velocity to zero. Use computational software to determine for any pair of values of $\omega_x$ and $\omega_y$ less than or equal to 0.06 rad/s, which of the thrusters should be used and for how long each of them should be activated. Apply this program assuming $\omega$ to be (a) the angular velocity given in Prob. 18.33, (b) the angular velocity given in Prob. 18.34, (c) $\omega = (0.06 \text{ rad/s})\hat{i} + (0.02 \text{ rad/s})\hat{k}$, (d) $\omega = -(0.06 \text{ rad/s})\hat{i} - (0.02 \text{ rad/s})\hat{k}$.

![Diagram of probe](image)

**Fig. P18.C2**
18.C3 A couple $M_0 = (0.03 \text{ lb} \cdot \text{ft})i$ is applied to an assembly consisting of pieces of sheet aluminum of uniform thickness and of total weight 2.7 lb, which are welded to a light axle supported by bearings at $A$ and $B$. Use computational software to determine the dynamic reactions exerted by the bearings on the axle at any time $t$ after the couple has been applied. Resolve these reactions into components directed along $y$ and $z$ axes rotating with the assembly. (a) calculate and plot the components of the reactions from $t = 0$ to $t = 2$ s at 0.1-s intervals. (b) Determine the time at which the $z$ components of the reactions at $A$ and $B$ are equal to zero.

18.C4 A 2.5-kg homogeneous disk of radius 80 mm can rotate with respect to arm $ABC$, which is welded to a shaft $DCE$ supported by bearings at $D$ and $E$. Both the arm and the shaft are of negligible mass. At time $t = 0$ a couple $M_0 = (0.5 \text{ N} \cdot \text{m})k$ is applied to shaft $DCE$. Knowing that at $t = 0$ the angular velocity of the disk is $\omega_1 = (60 \text{ rad/s})j$ and that friction in the bearing at $A$ causes the magnitude of $\omega_1$ to decrease at the rate of 15 rad/s$^2$, determine the dynamic reactions exerted on the shaft by the bearings at $D$ and $E$ at any time $t$. Resolve these reactions into components directed along $x$ and $y$ axes rotating with the shaft. Use computational software (a) to calculate the components of the reactions from $t = 0$ to $t = 4$ s (b) to determine the times $t_1$ and $t_2$ at which the $x$ and $y$ components of the reaction at $E$ are respectively equal to zero.
**18.C5** A homogeneous disk of radius 180 mm is welded to a rod AG of length 360 mm and of negligible mass which is connected by a clevis to a vertical shaft AB. The rod and disk can rotate freely about a horizontal axis AC, and shaft AB can rotate freely about a vertical axis. Initially rod AG forms a given angle \( \theta_0 \) with the downward vertical and its angular velocity \( \dot{\theta}_0 \) about the vertical. Use computational software (a) to calculate the minimum value \( \theta_m \) of the angle \( \theta \) in the ensuing motion and the period of oscillation in \( \theta \), that is, the time required for \( \theta \) to regain its initial value \( \theta_0 \). (b) to compute the angular velocity \( \dot{\phi} \) of shaft AB for values of \( \theta \) from \( \theta_0 \) to \( \theta_m \). Apply this program with the initial conditions (i) \( \theta_0 = 90^\circ \), \( \dot{\theta}_0 = 5 \text{ rad/s} \), (ii) \( \theta_0 = 90^\circ \), \( \dot{\phi}_0 = 10 \text{ rad/s} \), (iii) \( \theta_0 = 60^\circ \), \( \dot{\phi}_0 = 5 \text{ rad/s} \). [Hint: Use the principle of conservation of energy and the fact that the angular momentum of the body about the vertical through A is conserved to obtain an equation of the form \( \dot{\theta}^2 = f(\theta) \). This equation can be integrated by a numerical method.]

**18.C6** A homogeneous disk of radius 180 mm is welded to a rod AG of length 360 mm and of negligible mass which is supported by a ball-and-socket joint at A. The disk is released in the position \( \theta = \theta_0 \) with a rate of spin \( \dot{\psi}_0 \), a rate of precession \( \dot{\phi}_0 \), and a zero rate of nutation. Use computational software (a) to calculate the minimum value \( \theta_m \) of the angle \( \theta \) in the ensuing motion and the period of oscillation in \( \theta \), that is, the time required for \( \theta \) to regain its initial value \( \theta_0 \). (b) to compute the rate of spin \( \dot{\psi} \) and the rate of precession \( \dot{\phi} \) for values of \( \theta \) from \( \theta_0 \) to \( \theta_m \), using 2\(^\circ\) decrements. Apply this program with the initial conditions (i) \( \theta_0 = 90^\circ \), \( \dot{\psi}_0 = 50 \text{ rad/s} \), \( \dot{\phi}_0 = 0 \), (ii) \( \theta_0 = 90^\circ \), \( \dot{\psi}_0 = 0 \), \( \dot{\phi}_0 = 5 \text{ rad/s} \), (iii) \( \theta_0 = 90^\circ \), \( \dot{\psi}_0 = 50 \text{ rad/s} \), \( \dot{\phi}_0 = 5 \text{ rad/s} \), (iv) \( \theta_0 = 90^\circ \), \( \dot{\psi}_0 = 10 \text{ rad/s} \), \( \dot{\phi}_0 = 5 \text{ rad/s} \), (v) \( \theta_0 = 60^\circ \), \( \dot{\psi}_0 = 50 \text{ rad/s} \), \( \dot{\phi}_0 = 5 \text{ rad/s} \). [Hint: Use the principle of conservation of energy and the fact that the angular momentum of the body is conserved about both the Z and z axes to obtain an equation of the form \( \dot{\theta}^2 = f(\theta) \). This equation can be integrated by a numerical method.]
The Wind Damper inside of Taipei 101 helps protect against typhoons and earthquakes by reducing the effects of wind and vibrations on the building. Mechanical systems may undergo free vibrations or they may be subject to forced vibrations. The vibrations are damped when there is energy dissipation and undamped otherwise. This chapter is an introduction to many fundamental concepts in vibration analysis.